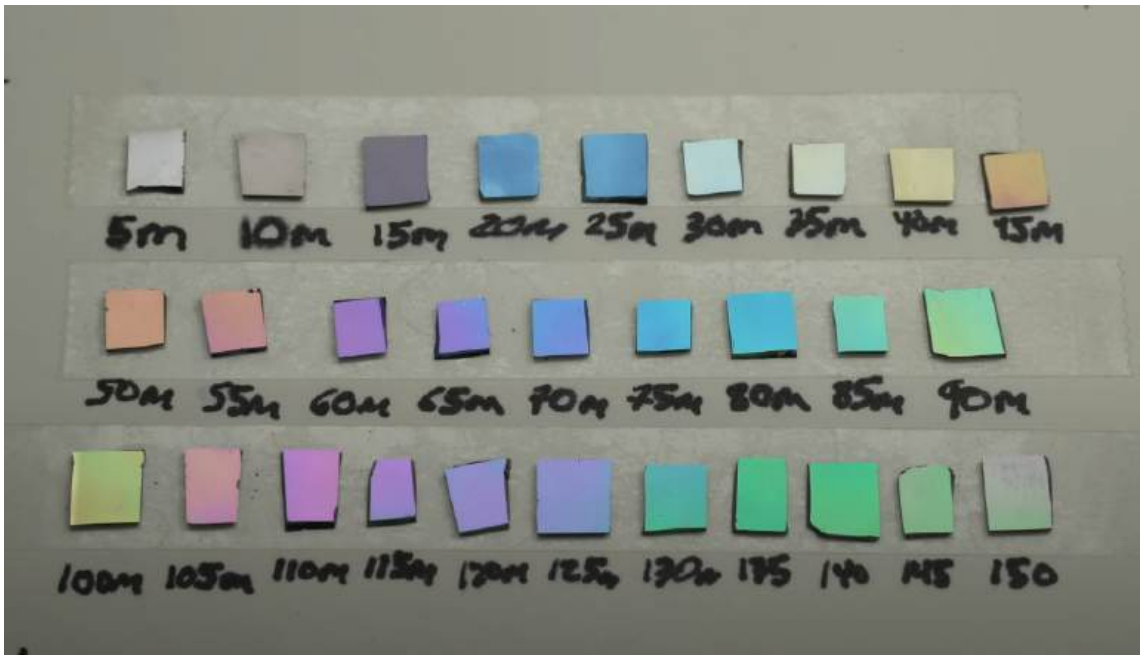
# T1. Thermal oxidation

Thermal oxidation is the process of creating a very thin (tens/hundreds of nanometres) film of silicon dioxide $SiO_2$ on the surface of silicon Si at high temperatures. This technology is an essential part of the manufacturing process of integrated circuits and MOSFETs (metal-oxide-semiconductor), which form the basis of modern electronics. If you place silicon Si in an aerobic environment, the oxidation process of $Si + O_2 \rightarrow SiO_2$ will begin. For the description of crystalline silicon and flat oxide films of sufficient thickness (starting from ten nanometres), the Dill-Grove model (1965) does well, but a detailed description of the growth mechanism in other conditions is still an unsolved scientific problem.

Silicon is opaque to visible light, while its dioxide is transparent. These optical properties, combined with the fact that the dioxide film has a thickness of tens of nanometres, create ideal conditions for very precise optical control of the thickness of the $SiO_2$ layer.



The change in the visible color of the silicon substrate at different oxidation times.
"Growing Colorful Oxide Layers on Silicon" by ProjectsInFlight on YouTube.

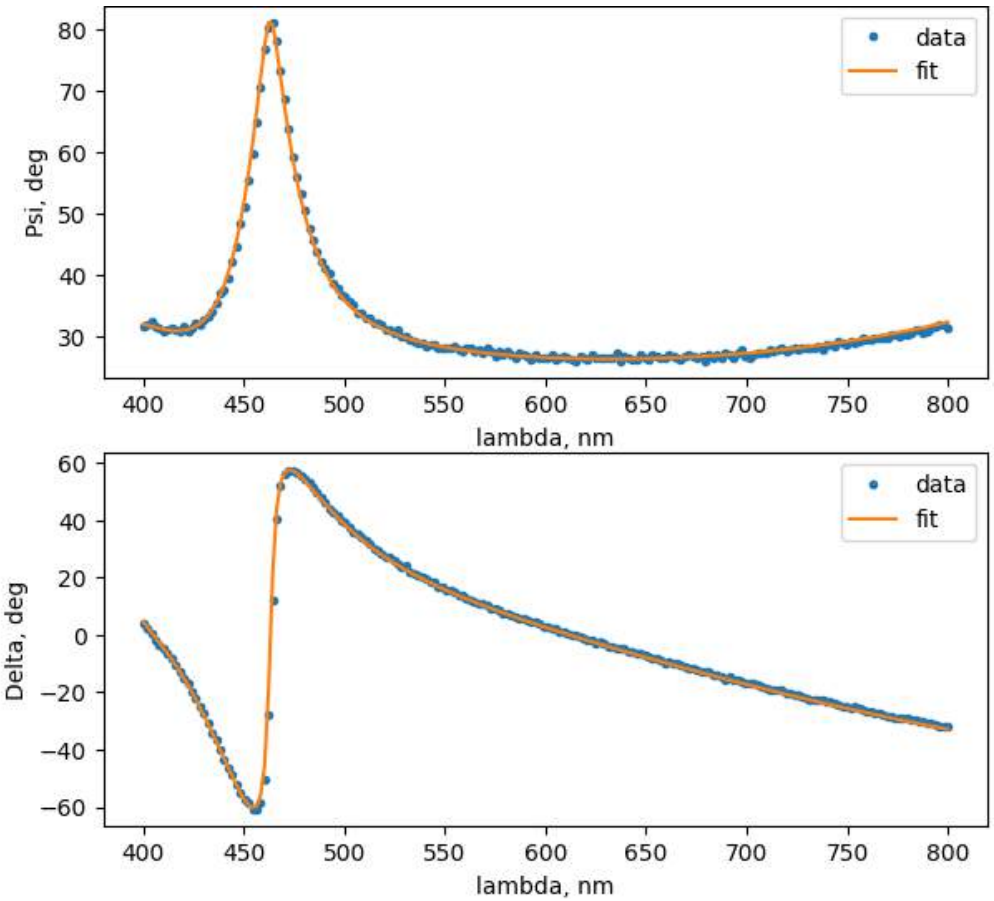
We will investigate the film growth by using ellipsometry, a powerful technique for studying the optical properties of thin films. It is based on a comparison of the amplitude coefficients of reflection of $s$-polarized and $p$-polarized waves (from the German "senkrecht" – perpendicular, "parallel" – parallel) from the surface of the film. The measurement results are represented by ellipsometric angles $\Psi$, $\Delta$:

$$\frac{r_p}{r_s} = \tan\Psi \cdot e^{i\Delta},$$

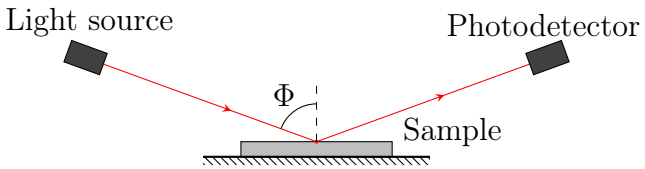
where the value of $\Psi$ corresponds to the ratio of amplitudes upon reflection of $p$- and $s$-polarized waves, and the value of $\Delta$ corresponds to the phase difference between them.

Ellipsometry measures precisely the relative difference between waves (ratio of amplitudes and phase difference) with different polarizations, so it is very resistant to various fluctuations in the power of the radiation source and the sensitivity of the receiver. Ellipsometry also makes it possible to measure the characteristics of a sample without direct mechanical contact with it, which allows measurements to be carried out in real time and ***in situ*** (from lat. – "on the spot", directly).

During the entire task, the wavelength of the incident wave is marked as $\lambda$. In the optical range, substances do not have significant magnetic qualities, therefore, all materials under consideration have $\mu = 1$.




Example of an ellipsometric spectrum $\Psi, \Delta$




Optical scheme of the ellipsometer: a light source, a Photodetector, and a test sample.

**Attention!** Place the files you want to attach to the report in the "report" folder.

**Attention!** It is enough to make only **A7** to moving on to part B.

# Part A. Theoretical principles of ellipsometry

Maxwell's equations in a non-magnetic dielectric have the form

$$\operatorname{div} \vec{D} = 0 \qquad \operatorname{rot} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\operatorname{div} \vec{B} = 0 \qquad \operatorname{rot} \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$

For a dielectric:

$$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$$

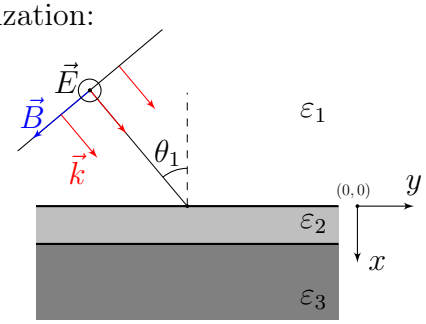
It follows from these equations that the normal components of the vectors $\vec{D}$ and $\vec{B}$ as well as the components of the vectors $\vec{E}$ and $\vec{B}$ directed along the surface do not change at the dielectric-dielectric boundary.

In a plane wave the wave vector, the electric field vector and the magnetic field vector $\vec{k}$, $\vec{E}$ and $\vec{B}$ are pairwise perpendicular and right-handed orientated. Also hold the following relations:

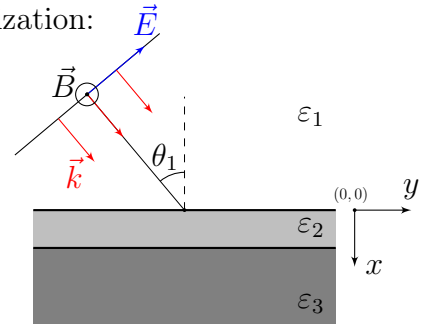
$$\vec{B} = \frac{1}{\omega}[\vec{k} \times \vec{E}], \qquad |\vec{k}| = \sqrt{\varepsilon}\omega/c$$

where $\omega$ is the frequency of the wave. The speed of light $c = 1/\sqrt{\varepsilon_0\mu_0}$. As a part of this task, we will consider the interaction of a plane wave with layered structures and develop a general approach for the theoretical description of such a system. Let a film with a thickness $d$ be made of a material with dielectric permittivity $\varepsilon_2$ be located between two semi-infinite dielectrics with dielectric permittivity $\varepsilon_1$ and $\varepsilon_3$ respectively.

s-polarization:




p-polarization:




The incidence of the $s$-polarized and $p$-polarized wave on the film

When a incident wave falls on the film, a reflected wave occurs in the medium $\varepsilon_1$, a passing wave in the medium $\varepsilon_3$ and a superposition of waves going in different directions in the medium $\varepsilon_2$. In $s$-polarization, the direction of the electric field is fixed, so we can proceed to consider scalar complex amplitudes:

$$\begin{cases} \tilde{E}_1 = \mathcal{E}e^{ik_{1x}x+ik_{1y}y} + r\cdot\mathcal{E}e^{-ik_{1x}x+ik_{1y}y} \\ \tilde{E}_2 = a\cdot\mathcal{E}e^{ik_{2x}x+ik_{2y}y} + b\cdot\mathcal{E}e^{-ik_{2x}x+ik_{2y}y} \\ \tilde{E}_3 = t\cdot\mathcal{E}e^{ik_{3x}(x-d)+ik_{3y}y} \end{cases}$$

In $p$-polarization, the direction of the magnetic field is fixed, so it is easier to proceed to the consideration of scalar complex amplitudes of the magnetic field:

$$\begin{cases} \tilde{B}_1 = \mathcal{B}e^{ik_{1x}x+ik_{1y}y} + r\cdot\mathcal{B}e^{-ik_{1x}x+ik_{1y}y} \\ \tilde{B}_2 = a\cdot\mathcal{B}e^{ik_{2x}x+ik_{2y}y} + b\cdot\mathcal{B}e^{-ik_{2x}x+ik_{2y}y} \\ \tilde{B}_3 = t\cdot\mathcal{B}e^{ik_{3x}(x-d)+ik_{3y}y} \end{cases}$$

| | |
|---|---|
| **A1** | Show that $k_{1y} = k_{2y} = k_{3y}$. |

| | |
|---|---|
| **A2** | Find expressions for $k_{1x}$, $k_{2x}$ and $k_{3x}$ in terms of $\omega$, $c$, $\theta_1$, $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$. |

| | |
|---|---|
| **A3** | For a wave with $s$-polarization (the electric field is directed along the surface), from the boundary conditions for the tangential components of the fields, obtain a system of linear equations for the coefficients $r$, $A$, $b$ and $t$. |

| | |
|---|---|
| **A4** | Find the reflection coefficient. Bring the answer to the form:<br>$$r_s = \frac{A + Be^{2ik_{2x}d}}{AB + e^{2ik_{2x}d}}.$$<br>Express the real numbers $A$, $B$ in terms of $k_{1x}$, $k_{2x}$ and $k_{3x}$. |

For convenience, use the variables $\kappa_{ix} = \dfrac{k_{ix}}{\varepsilon_i}$ for $i \in [1, 3]$.

| | |
|---|---|
| **A5** | For a wave with $p$-polarization (the magnetic field is directed along the surface), from the boundary conditions for the tangential components of the fields, obtain a system of linear equations for the coefficients $r$, $A$, $b$ and $t$. |

| | |
|---|---|
| **A6** | Find the reflection coefficient. Bring the answer to the form:<br>$$r_p = \frac{A + Be^{2ik_{2x}d}}{AB + e^{2ik_{2x}d}}.$$<br>Express the real numbers $A$, $B$ in terms of $k_{1x}$, $k_{2x}$ and $k_{3x}$. |

We will perform preliminary calculations of the approximate shape of the ellipsometric spectrum of a thin layer of silicon dioxide on a silicon substrate. Until the end of Part A, we will neglect the optical dispersion, and assume $\varepsilon_{\text{Air}} = 1.00$, $\varepsilon_{\text{Si}} = 17.2 - 0.430i$, $\varepsilon_{\text{SiO}_2} = 2.13$. Note that silicon has an imaginary part of the dielectric constant responsible for absorption, due to which it is opaque. The use of complex permittivity does not change obtained expressions, if the trigonometric functions of the complex parameter are considered.

With the help of the A.py program, you can plot $\Psi$ and $\Delta$ for different angles of incidence $\theta_1$ and thicknesses $d$. At the input, the program receives the name Name of the series, the angle of incidence $\theta_1$ from the air (in degrees) and the thickness $d$ of the dioxide film in the form of a table written in the file A_in.txt .

There is an example of filling in a table in a file A_in.txt:

| Name | $\theta_1$,° | $d$, nm |
|---|---|---|
| ex1 | 50 | 40 |
| ex2 | 30 | 100 |
| ⋮ | ⋮ | ⋮ |

The input data is limited to $20\,° \le \theta_1 \le 80\,°$, $d \le 250$ nm. The name of the experiment should not contain spaces and tabs. The separator between the integer and fractional parts of numbers is a dot. The separator between the columns is a tab.

| | |
|---|---|
| **A7** | Using the program A.py plot the dependence of $\Psi$ and $\Delta$ on $\lambda$ at different angles of incidence $\theta_1$ and thicknesses $d$. Conduct a quantitative study and determine at which angle of incidence $\theta_{\text{opt}}$ the ellipsometric angle $\Psi$ is most sensitive (the highest value $\lvert\Delta\Psi/\Delta d\rvert$) to changes in $d$ in the range from 10 nm to 50 nm. Find the value of $\theta_{\text{opt}}$ with an accuracy of one degree.<br>Since B3, use $\theta_{opt}$ as $\Phi$ to increase accuracy. |

# Part B. Kinetics of oxidation

In this part, the angle of incidence of the beam on the film will be indicated by $\Phi$.

Together with a colleague, you are studying the deposition of thin oxide films on the surface of silicon during prolonged calcination in the dry air. Real silicon Si and silicon dioxide $SiO_2$ have optical dispersion, so the graphs $\Psi, \Delta$ from $\lambda$ that will be offered to you in this part may slightly differ from those that you built in the part **A7**.

**Attention!** The optical dispersion of Si and $SiO_2$ is automatically embedded in all programs that you will work with in this part.

Your colleague was working late on the ellipsometer yesterday and forgot to write down the value of the angle $\Phi$. He measured the thickness of the oxide film on two samples and transmitted the measurement results to you in files B1.txt and B2.txt.

| | |
|---|---|
| **B1** | Using the program B1.py find by hand the value $\Phi$ and the thickness $d$ of $SiO_2$ film. The program B1.py reads values from a file B1.txt automatically and outputs an approximating graph $\Psi, \Delta$ from $\lambda$, for the values of the angle $\Phi$ entered by you in degrees and the thickness $d$ of the dioxide film in nanometers. The input data is limited: $20\,^\circ \le \Phi \le 80\,^\circ$, $d \le 250$ nm. |

| | |
|---|---|
| **B2** | Using the program B2.py find by hand the value $\Phi$ and the thickness $d$ of $SiO_2$ film. The program works similarly to the previous **B1**. |

For a detailed study of the oxidation rate, you can get ellipsometric data from your colleague, measured from a real-time oxidizing silicon substrate. At the initial moment $t = 0$, he puts pure silicon inside the furniture with a temperature $T$, and every 2 min mesures the spectrum $\Psi, \Delta$ from $\lambda$ at a fixed angle $\Phi$, which you tell him.

In the file B_in.txt you specify the name of the experiment, the angle $\Phi$ in degrees, the temperature $T$ in degrees Celsius, and the duration of the experiment in minutes in the form of a table:

| Name | $\Phi$,° | $T$,° C | $t$, min |
|---|---|---|---|
| test | 50 | 1000 | 40 |
| ⋮ | ⋮ | ⋮ | ⋮ |

The input data is limited as $20\,^\circ \le \Phi \le 80\,^\circ$, $800\,^\circ\text{C} \le T \le 1200\,^\circ\text{C}$, $t \le 200$ min. The name of the experiment should **not** contain spaces and tabs. The separator between the integer and fractional parts of a number is a dot. The separator between the columns is a tab. Next, you run the program gen_data.exe which simulates your colleague conducting an experiment in accelerated mode. You can interrupt the execution of this program at any time, and the results of the measurements already performed will be saved on your computer. You can automatically process the measurement data using the program B.py which uses the same file B_in.txt . This program uses the results of the work gen_data.exe, selects the value of $d$ for each point and calculates the statistical error of this value. Keep in mind that processing may take several minutes. The results of the program B.py are located in the /res folder in the form of tables and graphs $d(t)$. An example of an automatically generated program B.py tables in the /res folder:

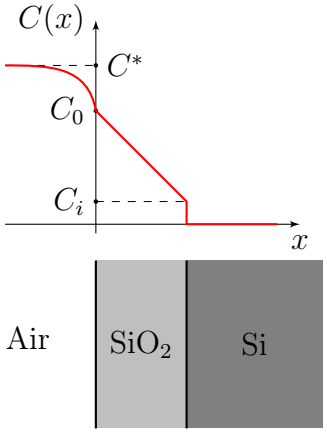
| $t$, min | $d$, nm | $\Delta d$, nm |
|---|---|---|
| 2 | 17.3 | 0.3 |
| ⋮ | ⋮ | ⋮ |

| | |
|---|---|
| **B3** | Get the data for the dependence of the film thickness $d$ of silicon dioxide on the time $t$ during oxidation at a temperature of $t = 1000\,^\circ\text{C}$ for 120 min. Attach the graph obtained by the program B.py to the report. |

Now let us consider the theoretical aspects of thermal oxidation. According to the Deal-Grove model, three processes are responsible for the rate of increase in the thickness of the silicon oxide layer:

1. diffusion of oxygen to the surface $SiO_2$;
2. diffusion of oxygen through the layer $SiO_2$;
3. oxidation reaction $Si + O_2 \rightarrow SiO_2$ at the interface $SiO_2$/Si.

Thus, the concentration of oxygen $C$ (dimension mol/m$^3$) changes as it moves deeper into the sample. In the air, far away from the substrate, the oxygen concentration is $C^*$; inside the silicon dioxide layer, the concentration decreases linearly from $C_0$ to $C_i$; inside silicon, the oxygen concentration is zero.




Let us consider a model for describing these processes. Specific (i.e. per unit surface area) the flow rates of each of the three processes $F_1$, $F_2$, $F_3$ (units of measurement — mole/(c $\cdot$ m$^2$)) we define as follows:

1. The diffusion rate is $F_1 = h(C^* - C_0)$, where $h$ is a certain coefficient characterizing diffusion in the gas phase.
2. Counting the diffusion rate $F_2 = \frac{D}{d}(C_0 - C_i)$, where $D$ is the diffusion coefficient of oxygen in the dioxide, depending on the temperature $T$, and $d$ is the thickness of the silicon dioxide film.
3. The rate of the oxidation reaction $F_3 = kC_i$, where $k$ is the constant of the rate of the oxidation reaction, depending on the temperature $T$.

| | |
|---|---|
| **B4** | Express the growing velocity $v$ of film thickness change in terms of the oxidation reaction rate $F_3$, the molar mass $\mu_{SiO_2}$ and the density $\rho_{SiO_2}$ of silicon dioxide. Assume that the air-dioxide boundary does not move. |

| | |
|---|---|
| **B5** | Assuming the problem to be quasi-stationary (the establishment times of all processes are much less than the film rise time), express $C_0$ and $C_i$ in terms of $C^*$, the velocity constants $h$, $k$, $D$ and the film thickness $d$. |

| | |
|---|---|
| **B6** | Get the differential equation for the thickness $d$ of the dioxide film and get its solution with the initial condition $d(t_0) = d_0$ in the form<br>$$d^2 + Ad = B(t + \tau),$$<br>where $\tau$ contains information about the initial conditions, and the constants $A$ and $B$ are related to the kinetics of oxidation. Express $A$ and $B$ in terms of $h$, $k$, $D$, $C^*$, $\mu_{SiO_2}$ and $\rho_{SiO_2}$. |

Experiments show that blowing with air (changing the profile $C(x)$ above the surface), doesn't change the rate of the process. Therefore we will consider $h \gg k$.

At the beginning of the oxidation process, the stationary model is not applicable, so it is used starting from the minimum thickness $d_0 = 25$ nm. Enter into the program B7.py the data from the file B_in.txt and get the parameters $B$ and $B/A$ automatically. If the maximum film thickness obtained in the experiment is less than $d_0$, then the program B7.py outputs "Final layer is too thin to approximate".

If the program returns the error "FileNotFoundError", run the files gen_data.exe and B.py sequentially with the input data you need, previously entered in the B_in.txt.

| **B7** | Get the values of the parameters $B$ and $B/A$ for five different temperatures $T$. Save the results into a file B8_in.txt in the form of a table. |
|---|---|

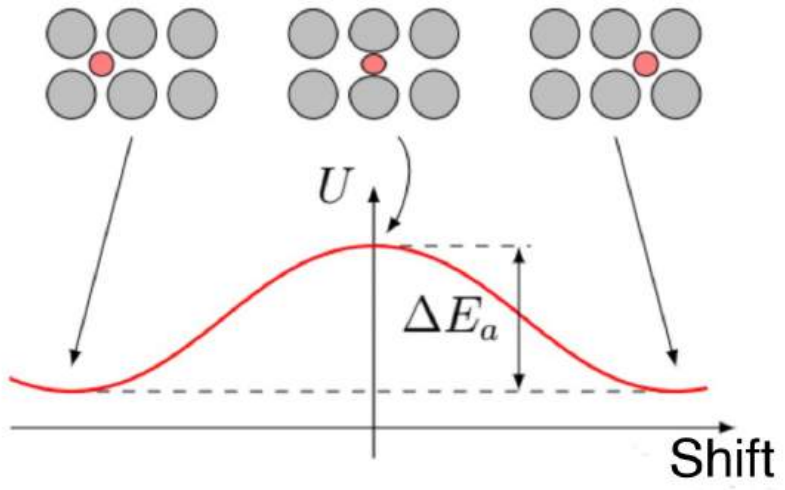
| $T$,° C | $B$, nm$^2$/min | $\Delta B$, nm$^2$/min | $B/A$, nm/min | $\Delta(B/A)$, nm/min |
|---|---|---|---|---|
| 1000 | 174 | 4 | 1.21 | 0.02 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

The dependence of the reaction rate constant $k$ on temperature $T$ is associated with the energy $\Delta E_b$ of breaking the bond Si − Si in a silicon crystal, and then the relation between them:

$$k \propto e^{-\Delta E_b/k_B T}.$$

The diffusion coefficient in solids is also related to temperature via the so-called activation energy $\Delta E_a$, and the quantitative dependence has the form:

$$D \propto e^{-\Delta E_a/k_B T}$$




A qualitative diffusion scheme in a solid and the meaning of energy $\Delta E_a$.

Using the program B8.py you can plot graphs $\ln(B \cdot \text{min/nm}^2)$ from $1/T$ and $\ln(B/A \cdot \text{min/nm})$from $1/T$ and determine the parameters of the automatically drawn lines.

Boltzmann constant $k_B = 1.38 \cdot 10^{-23}$ J/C, elementary charge $e = 1.60 \cdot 10^{-19}$ C.

| **B8** | Determine the values of $\Delta E_b$ and $\Delta E_a$ in electron volts. Attach the linearized graphs to the report. |
|---|---|

# T2. Memristors and NVRAM

## Introduction. Non-volatile random access memory

**NVRAM** (Non-Volatile Random Access Memory) is a type of random access memory capable of storing information in the absence of electrical power. The ability to independently access information in any memory cell at any time and to store data without constant power supply opened huge technological capabilities at the time of its invention, and we still use it today (Fig. 1).

There are several types of NVRAM. The most popular of them are based on semiconductor elements, and therefore are difficult to consider. In this task, we will examine both a promising realization of NVRAM – ReRAM (Resistive Random Access Memory) based on memristors, and a realization that have already almost disappeared into the past – ferroelectric memory (FeRAM, Ferroelectric Random Access Memory) based on segnetoelectrics.

Fig. 1. NVRAM in your pocket

## Part A. Memristor

In 1971, Prof. Leon O. Chua published a theoretical justification for the possibility of a **memristor** – an electrical element in which the relationship between voltage and current depends on the total charge flowing through the element.

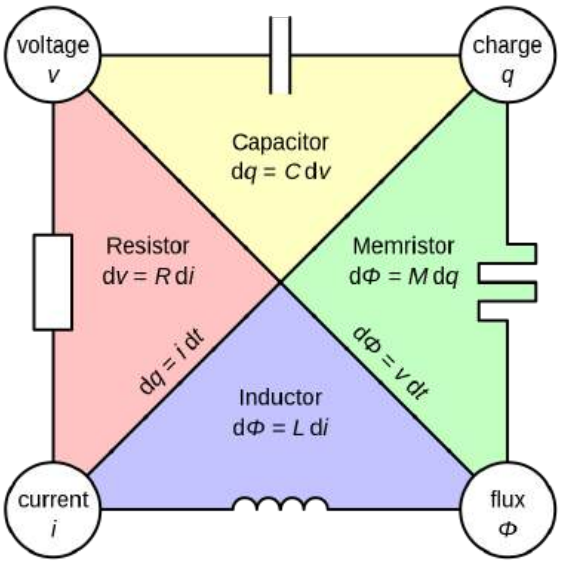



Fig. 2. Schematic depiction of the relationship between different electromagnetic quantities in various electronic components. Usually, this scheme is cited as part of the theoretical justification of the possibility of the existence of a memristor

Memristor-based computing systems are currently attracting serious attention of researchers in the field of artificial intelligence, as they work on principles similar to the neurons of the human brain and can potentially achieve the same results with much lower energy consumption and relative architecture simplicity.

For a long time, the memristor existed only "on paper", until in 2008 Hewlett-Packard presented the first working prototype of a memristor. In this prototype, the change in resistance depending on the flowing charge was achieved due to electrochemical reactions occurring in the contact area of two different materials (Fig. 3).

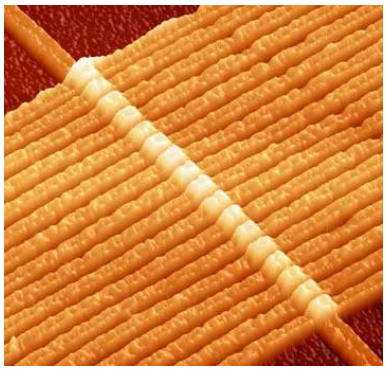


Fig. 3. Atomic force microscope topographical scan of memristor prototype created by HP

Since electrochemical reactions are reversible, the resistance of this prototype memristor tends towards its equilibrium value in the absense of current. For the simplest theoretical description of such a memristor, we can use the **Williams–Strukov equations**:

$$\begin{cases} R(x) = R_{\text{off}}(1-x) + R_{\text{on}}x \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{\beta}I - \alpha x \end{cases}$$

Here $x$ is some parameter determined by the internal state of the system, $\alpha, \beta > 0$ are known quantities, $I$ is current through the memristor, $R_{\text{off}}$ and $R_{\text{on}} = rR_{\text{off}}$ ($r \gg 1$) are the resistances of the memristor in the "off" and "on" states, respectively.

Let's first consider the evolution of the memristor state under a constant voltage $U > 0$. Initially, the memristor is "off" (i.e. $x = 0$).

| | |
|---|---|
| **A1** | Obtain a differential equation on $x(t)$. Express your answer in terms of $U$, $R_{\text{off}}$, $r$, $\alpha$, and $\beta$. |

For convenience, let's introduce $\xi \equiv \dfrac{r-1}{\beta R_{\text{off}}}$. We will assume that the memristor is on if the value of its internal parameter $x$ differs from the equilibrium one by no more than 10%. For simplicity in all the remaining points of this part, **work in the limit of large voltages** $U$.

| | |
|---|---|
| **A2** | Simplifying the equation from **A1** in this limit, find the minimum value of $x_0$ at which the memristor can be considered to be on. Express your answer in terms of $\xi$, $U$, $\alpha$, and $r$. |

| | |
|---|---|
| **A3** | Find the time $\tau$ required to switch the memristor from the initial off state to the on state. Express the answer in terms of $\alpha$. |

| | |
|---|---|
| **A4** | Find the minimal energy $Q$ requires to switch the memristor from the initial off state to the on state. Express your answer in terms of $U$, $\alpha$, $\xi$, and $R_{\text{off}}$. |

# Part B. Memristor hysteresis

If an an oscillating voltage is applied to the memristor, its resistance will also oscillate due to fluctuations in current and charge flowing through it. Because of this, at the same voltage, the current through the memristor will depend on whether the voltage increases or decreases. In $I - U$ coordinates, this behavior will look like hysteresis loops (Figure 4). It turns out that by measuring the characteristics of these loops it is possible to calculate some important parameters of the memristor, which we will do in this part.

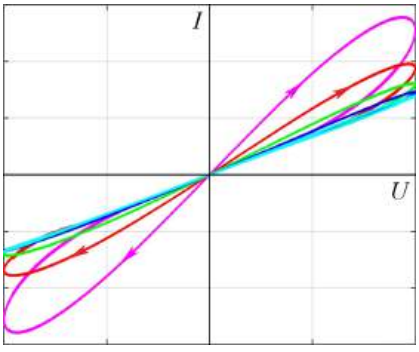



Fig. 4. Memristor hysteresis loops on different frequencies

Suppose an oscillating voltage $U\cos\omega t$ is applied to the memristor, and $U > 0$.

| | |
|---|---|
| **B1** | Find $x(t)$ in the first-order approximation by $\dfrac{U}{\beta R_{\text{off}}\sqrt{\omega^2+\alpha^2}} \ll 1$. From here, find (also to a first-order approximation), $\Delta I(\varphi)$ – the absolute difference in currents on different branches of the hysteresis loop at the same voltage $U\cos\varphi$. Express your answers in terms of $U$, $R_{\text{off}}$, $\alpha$, $\beta$, $r$, and $\omega$. |

Let's introduce the **hysteresis loop width** as:

$$s = \max_{\varphi} \frac{\Delta I(\varphi)}{2I_0},$$

where $I_0$ is the amplitude of the current oscillations.

| | |
|---|---|
| **B2** | Express $s$ in terms of $\omega$, $\alpha$, $U$, and $\xi$. |

Since the hysteresis curve width is easy to measure, it can be used to determine the parameters of the memristor. The table below shows the dependence of $s$ on the frequency $f$ of the voltage applied to the memristor at voltage $U = 1\ V$.

| | | | |
|---|---|---|---|
| $f$, Hz | 100 | 250 | 500 |
| $s$, $10^{-3}$ | 47.8 | 54.1 | 36.7 |

Also, the resistance of the memristor $R_{\text{off}} = 300\,\Omega$ was measured in the off state.

| | |
|---|---|
| **B3** | From this data, **calculate** $\alpha$ and $\xi$ with three significant figures. |

# Part C. ReRAM energy efficiency

The information is written to the prototype memristor element under a voltage of $U = 5\ V$.

| | |
|---|---|
| **C1** | **Calculate** with two significant figures the time $\tau$ in which the write takes place. |

| | |
|---|---|
| **C2** | **Calculate** with two significant figures the minimum amount of energy $Q$ required to write one bit of information. |

As you can see, the researchers at HP are still pretty far from perfect...

# Part D. Segnetoelectrics and FeRAM

In most materials, polarization is identically equal to zero in the absence of an external field. However, there is a class of materials called **segnetoelectrics** that remain polarized even when no field is applied. The ability to store information in the direction of polarization and to read and rewrite it easily was the basis for the creation of ferroelectric memory (FeRAM). In this part of the task we will investigate these properties of segnetoelectrics in detail.

The volume energy density $\mathcal{W}$ in a segnetoelectric depends on the polarization $P$ and the external electric field $E$ as:

$$\mathcal{W}(P) = -aP^2 + bP^4 - PE,$$

where $a, b > 0$ are material-specific quantities. The equilibrium values of polarization are determined by local minima of $\mathcal{W}(P)$. In the absence of an external field, there are two equilibrium states of a segnetoelectric with the same energy, transitions between which are impossible, which is used for information storage.

Consider a segnetoelectric material that is initially in the state with negative $P$, and at some moment the external electric field is switched on and starts to increase. Then, when the field reaches some critical value $E_{\text{cr}}$, the local minimum with smaller polarization disappears and the system jumps to the only remaining equilibrium state, dissipating at this moment the energy $\mathcal{Q} = -\Delta\mathcal{W}$ per unit volume in the form of heat. If the external field is then slowly turned off, the system will move to an equilibrium state with positive $P$ (that is, its state will change). This is used to record information.

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| **D1** | Find at what critical field $E_{\text{cr}}$ one of the equilibrium states of the segnetoelectric in the above situation disappears. What is the polarization $P_{\text{cr}}$ in this state just before the "disappearance"? Express your answers in terms of $a$ and $b$.<br>**Hint:** consider the second derivative of energy density at the moment of "disappearance". |

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| **D2** | What polarization $P_{\text{after}}$ will the segnetoelectric have right after the "jump"? Express your answer in terms of $a$ and $b$. |

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| **D3** | Find the dissipated energy $\mathcal{Q}$ per unit volume. Express your answer in terms of $a$ and $b$. |

The first commercial FeRAM samples used lead zirconate titanate, for which $a = 1.9 \cdot 10^5 \ \dfrac{\text{J} \cdot \text{m}}{\text{C}^2}$ and $b = 1.7 \cdot 10^3 \ \dfrac{\text{J} \cdot \text{m}^5}{\text{C}^4}$. The memory cells were fabricated on a $l = 350$ nm process, which can be used to estimate the cell volume.

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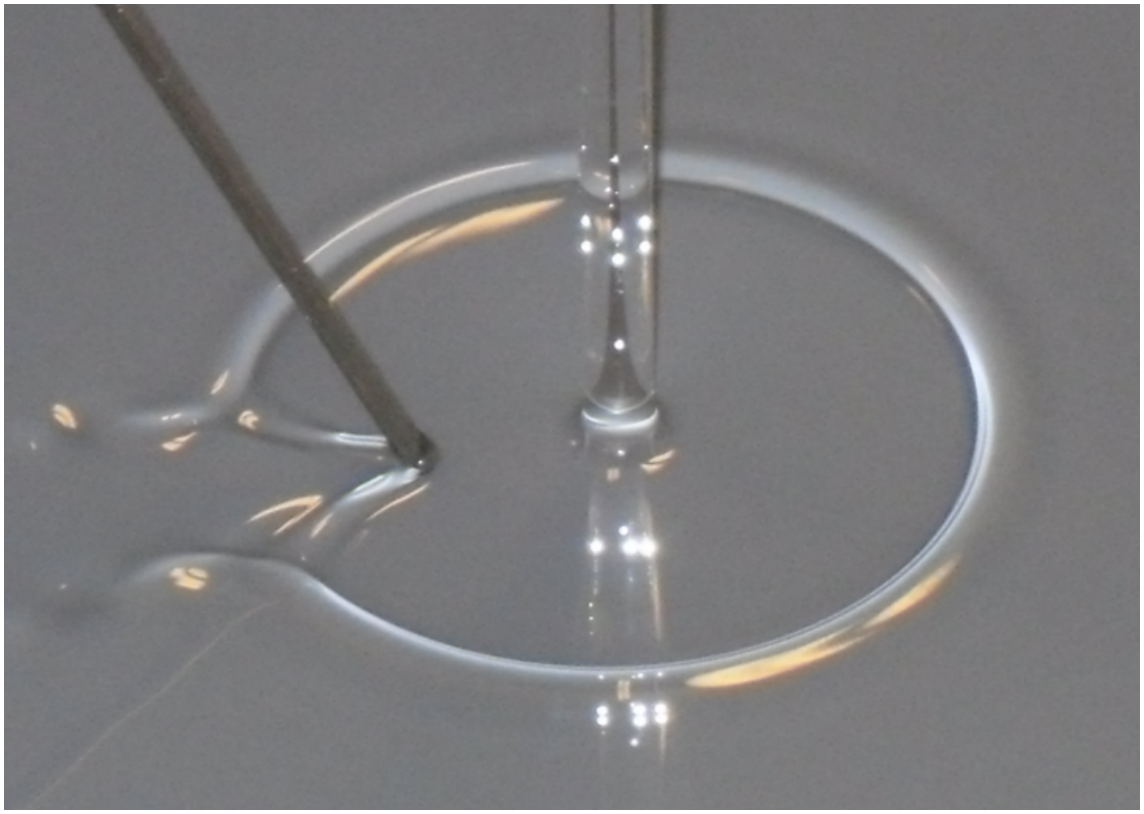
| **D4** | What amount of energy $Q_{1\ \text{TiB}}$ is required to write 1 TiB $= 2^{43}$ bits of data to FeRAM? Express your answer in terms of $a$, $b$, and $l$. Calculate this quantity with two significant figures. |

# T3. Hydraulic jump

A hydraulic jump is a phenomenon in which the rapid flow of fluid slows down abruptly, resulting in an elevation of the fluid level within the stream. This phenomenon can occur naturally in the flow of a river or canal. Additionally, it is utilised in the construction of dams to decelerate the speed of water flow. Furthermore, hydraulic jumping can be observed in domestic settings. For instance, when a jet of water strikes the surface of a sink, a circular formation with a thin layer of fast-flowing water is formed around it. At a certain distance from the jet, the water level rises, which is hydraulic jumping. The problem requires you to describe this phenomenon using conservation laws.



Hydraulic jump around a fluid jet falling on a flat surface. From the work of G. Jannes, R. Piquet, P. Maïssa, C. Mathis, and G. Rousseaux Phys. Rev. E 83, 056312

The entire problem considers the flow of water, which can be considered an incompressible fluid with a density $\rho = 1.0 \cdot 10^3$ kg/m$^3$. The acceleration of free fall is $g = 9.8$ m/$s^2$.

If energy losses are neglected, the Bernoulli equation is satisfied for stationary (time-independent) fluid flow along any current tube. This can be expressed as follows:

$$E = \frac{P}{\rho} + \frac{v^2}{2} + gz = const$$

In this equation, $P$ represents the pressure, $v$ is the flow velocity, and $z$ denotes the height of the fluid volume under consideration. The value $E$ will be referred to as the specific energy of the given fluid element. The Froude number is employed as a dimensionless parameter to characterise the flow of a fluid, taking into account the force of gravity. This is expressed as follows:

$$Fr = \frac{v}{\sqrt{gd}},$$

where $v$ is the characteristic flow velocity, $d$ is the flow depth.

It should be noted that throughout the entirety of the problem, the atmospheric pressure does not affect the fluid flow. Consequently, its contribution to the Bernoulli equation and to the expressions for forces can be ignored.

# Part A. Flow without energy loss

In this part, we will consider the flow of a fluid along a rectangular canal of width $b$. The fluid motion can be considered steady-state, meaning that the velocity at a given point in the canal is independent of time. The water flow through the canal (the volume of water flowing through the channel cross-section per unit time) is $Q$. It is assumed that the canal bottom is flat and that the flow velocity is uniform throughout the canal cross-section. Atmospheric pressure does not need to be taken into account.

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| **A1** | The depth of water in the canal is $d$. Find the specific energy of the flow $E$. The height of the canal, $z$, should be count from the bottom. The answer should be expressed in terms of the following variables: $Q$, $b$, $d$, and $g$. |

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| **A2** | Determine the critical water depth, $d_c$, at which the specific energy $E$ is minimal. Express it in terms of $Q$, $b$, and $g$. |

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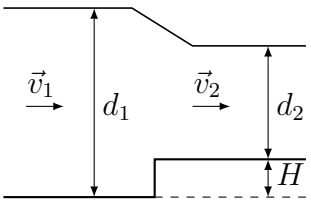
| **A3** | Determine the fluid velocity $v_c$ and Froude number $Fr_c$ in the case of critical depth. The answer should be expressed in terms of $Q$, $b$, $g$. |




Suppose now that at some point the bottom of the canal rises to a height $H$ and its width remains unchanged. Let $d_1$ and $v_1$ be the water depth and flow velocity before the change of the depth, $d_2$, $v_2$ – after the change.

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| **A4** | Write two equations following from the laws of conservation of mass and energy of water that relate these quantities. These equations may also include $g$ and $H$. It is assumed that at all points of the flow the flow is laminar and Bernoulli's law is fulfilled |

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| **A5** | The equations written in the previous task can not be solved exactly. Therefore, we will assume that the height $H$ is small. Consequently, $\Delta v = v_2 - v_1 \ll v_1$ and $\Delta d = d_2 - d_1 \ll d_1$ are small changes in velocity and depth of flow. In all calculations, we limit ourselves to first-order contributions of $H$, $\Delta v$, and $\Delta d$. Determine the values of the ratio $\Delta v/v_1$ and the ratio $\Delta d/d_1$. |

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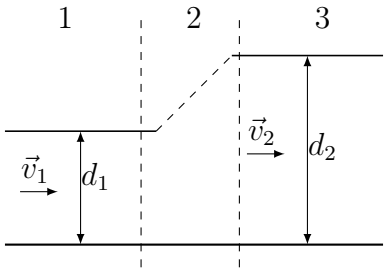
| **A6** | Indicate at which values of the Froude number of the flow before changing the depth of the canal the velocity increases and at which values the velocity decreases. |

# Part B. Theory of hydraulic jumping

In this part we will examine the hydraulic jump in detail. For this purpose, we will again consider the flow of fluid along a rectangular canal of constant width $b$. The flow area can be divided into three parts:

1. an initial part of fast flow of small depth, where velocity and depth are $v_1$ and $d_1$;
2. a transition part, the flow in which is highly turbulent and the depth changes and energy losses occur, so the Bernoulli equation cannot be used;
3. a part of slow flow of constant velocity $v_2$ and depth $d_2$.




To describe the hydrodynamic jump, let us select the volume of water in the canal in such a way that the left boundary of the selected region lies in part 1 and the right boundary lies in part 3. The law of conservation of mass and the law of momentum change can be applied to the selected volume.

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| **B1** | Find the hydrostatic pressure force $F$ acting on the selected volume. It can be demonstrated that the contribution of atmospheric pressure is zero and can be disregarded (proof is not required). It is assumed that the pressure dependence on height is the same as in a stationary fluid. The answer must be expressed in terms of $\rho$, $g$, $b$, $d_1$, $d_2$. |

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| **B2** | Write two equations following from the law of conservation of mass and the law of change of momentum that connect $v_1$, $v_2$, $d_1$, $d_2$. The answer may also include $g$, $b$, $\rho$. |

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| **B3** | Using the equations obtained, determine the ratio of the water depth after the jump to the depth before the jump $d_2/d_1$. Express the answer in terms of the Froude number of the flow before the jump $Fr$. |

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| **B4** | Find the difference of specific energies of the liquid before and after the jump $\Delta E = E_2 - E_1$. Express the answer in terms of $g$, $d_1$, $d_2$. |

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| **B5** | At what values of the Froude number is hydrodynamic jump possible? |

# Part C. Hydraulic jump in the sink

In this part, we will use our previous results to estimate the parameters of the hydrodynamic jump that can be observed in the sink. A complete theory would be required to determine the radius of the circle on which the jump occurs. However, it is necessary to take into account fluid viscosity and surface tension. Therefore, we will consider this radius as a given value.

Let a jet with a volumetric flow rate $Q = 3.0 \cdot 10^{-5}$ m$^3$/s hit a horizontal surface, the jet diameter just before contact with the surface $D = 1.0$ cm, the jet is perpendicular to the surface. The flow is symmetric about the symmetry axis of the jet. Consider that the law of conservation of energy is fulfilled in the fluid flow up to the moment of hydraulic jump, and the flow velocity does not depend on the distance from the jet.

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| **C1** | Find the velocity $v$ of fluid flow before the hydraulic jump. Give the formula and the numerical value. The answer should be expressed in terms of $Q$ and $D$. |

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| **C2** | How does the depth $d$ of water depend on the distance $r$ from the center of the jet? Express the answer in terms of $D$, $r$, find the numerical value when $r = r_c = 3$ cm (the radius at which the jump occurs). |

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| **C3** | Find the Froude number (numerical value) at the point where the hydrodynamic jump occurs. Determine how many times the water depth increases during the jump $d_2/d_1$. |