T1. Polycrystal diffusion

Instructions for working with the program can be found in the Appendix. It is recommended to read the text of the task and the Appendix completely before starting the measurements.

In the preparatory lectures we have already discussed diffusion in solids and its applications. This problem also discusses diffusion of impurities in solids, in particular, in polycrystals.

Part A. Temperature dependence of the diffusion coefficient (2.6 points)

This part of the problem discusses the diffusion of impurity particles within the crystal lattice of a substance.

The diffusion process in a crystal lattice is carried out in most "ordinary"substances by the jump of diffusive impurity atoms between neighboring lattice cells. The impurity particles spend most of their time in the vicinity of the point where the potential energy of interaction between the impurity particle and the lattice atoms is minimal. The vicinity of such points is called **inter-nodes**. Usually, the energy of impurity particles is low, so they can only jump between neighboring inter-nodes. For this purpose, an energy barrier of at least E_g must be overcome. The energy E_g is called the **activation energy**. It is known from statistical physics that the probability of such a transition is proportional to $\exp(-E_g/kT)$. Moreover, the jump time between neighboring internodes should be inversely proportional to the characteristic particle velocity. Thus,

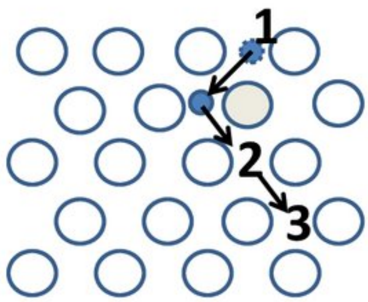


Figure 1. Inter-nodal diffusion of impurity atoms

$$D \propto v \exp(-E_g/kT),$$

where v is the rms velocity associated with the thermal motion of impurity particles.

A1	Write the expression for v in terms of T and physical constants. From this, obtain the law of proportionality $D(T)$.	**0.2**

A2	Write down how the rms displacement of a particle Δr_{sq}, diffusion coefficient D and simulation time t are related to each other in the **three-dimensional case**. *Note:* the particle motion along each of the three axes in random walk can be considered independent. The diffusion coefficient D in **one-dimensional case is** related to the cell size a and characteristic jump time τ as $D = a^2/2\tau$. *Note:* if you cannot complete this task, in the future, consider $\Delta r_{sq} = \sqrt{Dt}$.	**0.4**

The program for part **A** simulates the motion of impurity particles in the inter-nodes for a cubic lattice. Select the number of particles to be simulated, as well as the temperature and simulation time. The program will output the root mean square of the particle displacement and its error among the simulations. All program inputs and outputs are made in arbitrary units.

A2	Get the points to find $D(T)$ and linearize the resulting dependence. Find the energy E_g in arbitrary units with an accuracy of at least 2%. Explicitly state how you achieve this accuracy. Make measurements in $T \in [1; 10]$, $t \in [1; 10]$, and $N \in [10^2; 10^4]$.	**2.0**

In the remaining part of the problem, a solid of large dimensions with a flat boundary is modeled, on the surface of which some large number of impurity atoms are uniformly deposited. Over time, the impurity atoms diffuse into the substance. The result of the simulation is the concentration distribution c of the impurity particles in the solid depending on the penetration depth z of these particles.

Part B. Effective diffusion coefficient (2.4 points)

If all solids were completely homogeneous bodies at the microscopic level, the problem of diffusion inside them would be trivial. However, most materials in their structure are many small single crystals-grains, on the boundaries between which the atoms of matter are arranged chaotically. Although this state does not have minimum energy, at normal temperatures it is still the most probable from a thermodynamic point of view. In particular, almost all metals are polycrystalline in nature (e.g., it is this mechanism that ensures their plasticity, as it allows parts of the crystal lattice to move relative to each other).

Since the atoms at the grain boundary are arranged chaotically and much more "loose the movement of impurity atoms between them does not require such a large amount of energy as movement inside the lattice of a single crystal. Because of this, the diffusion coefficient at the grain boundary is much larger than inside them, which can noticeably affect the diffusion processes in polycrystal. The purpose of Parts **B** and **C** of this problem is to investigate the phenomena that occur during polycrystalline diffusion.

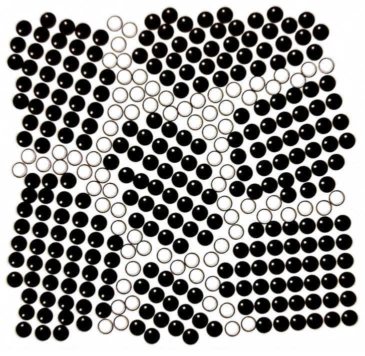


Figure 2. Characteristic structure of polycrystal

Diffusion processes in a polycrystal have three different regimes. They are called kinetic modes of type A, B, and C, respectively. The kinetic regime of type A is observed on the largest time scales, regime C – on the smallest. Mode B is intermediate between them. Only modes A and B are investigated in the problem. When the characteristic diffusion scale $\sim \sqrt{Dt}$ exceeds the grain size of the polycrystal, each atom of the "embedded"substance passes through a large number of grains. Due to this, diffusion proceeds in a single "front"and can be characterized by the effective diffusion coefficient D_{eff}, which depends on the coefficient of diffusion in grains D, at grain boundaries D_{gb} and relative volume δ occupied by the boundaries.

If the grain boundaries in a polycrystal were oriented exclusively along the diffusion, then the effective diffusion coefficient would be equal simply to the weighted average of D and D_{gb}:

$$D_{eff} = D + \delta(D_{gb} - D).$$

If the boundaries are oriented perpendicularly to the diffusion direction, then at $\delta \ll 1$ the effective diffusion coefficient **would practically not change**. In a real polycrystal, the grain boundaries are oriented arbitrarily. Because of this, the effective diffusion coefficient differs from that of the measures given above. It is very difficult to solve precisely the problem on the effective diffusion coefficient in such a structure, and in reality one can only give an estimate in the limiting case $g \ll 1$, $D_{gb}/D \gg 1$:

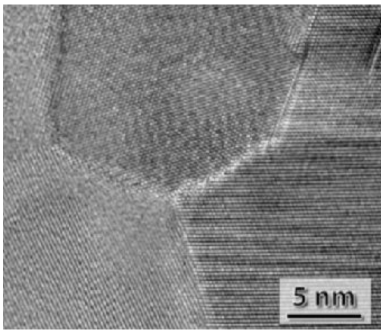


Figure 3. Image of a polycrystal made with atomic force microscope

$$D_{eff}/D = 1 + \alpha\delta(D_{gb}/D - 1).$$

Let us try to estimate α in some approximation and then find it experimentally. The easiest way to estimate α is to take $D \to 0$ at $D_{gb} = \text{const}$ and find D_{eff} for some simple three-dimensional configuration of the polycrystal. For example, let us take a polycrystal consisting of cubic single crystals of size a separated by boundaries of thickness $\delta a/3$. The effective diffusion coefficient can be estimated as the diffusion coefficient along one of the axes of the system (it will be the same along all three axes).

B1	What will be the coefficient α in such a structure? *Note:* It is not necessary to solve this task to proceed to numerical modeling.	**0.4**

Let's check the fairness of the estimation with the help of the program. The program simulates an arbitrary polycrystalline structure. The program in parts **B** and **C** takes as input values in relative units such that $a, D = 1$, and the time unit is D/a^2. You can adjust the diffusion coefficient at the boundary D_{gb}, the simulation time t and the relative volume of the boundaries δ.

Note: when performing the following paragraph, consider it known that the distribution of particle concentration c from the penetration depth z in the case of a homogeneous crystal with diffusion coefficient D_{eff} would have the following form:

$$c(z) = \frac{N_0}{4\sqrt{\pi D_{eff} t}} \exp\left[-\frac{z^2}{4D_{eff}t}\right].$$

B2	Simulate diffusion in a polycrystal with $t \in [2; 5]$, $D_{gb}/D \in [2; 5]$, and $\delta \in [0.005; 0.05]$, to obtain the experimental value of α. Compare with the result of **B1**. *Note:* As a result of the program operation, the dependence of the logarithm of the mean concentration $\ln \bar{c}$ of impurity particles on the square of the penetration depth z^2 is written to the file Bavg.csv. This dependence can be analyzed with the program B.nb.	**2.0**

Finally, it is worth noting that the coefficient α depends on the dimensionality of the space in which diffusion takes place.

Part C. Type B kinetics (5 points)

In the previous parts of the problem, we considered the diffusion type at which the characteristic scales of diffusion are much larger than the size of inhomogeneities in the polycrystal. However, this is not the only diffusion mode that can be observed in a polycrystal. If the diffusion scales are much larger than the grain boundary width of the polycrystal and much smaller than the size of the grains themselves, diffusion of kinetic type B will occur.

The main distinguishing characteristic of this type of diffusion is that the logarithm of the mean concentration $\bar{c}$ depends on the depth z not quadratically, but to the degree $6/5$. This value is not analytically derived, but it is so well observed in the experiment that the presence of such a dependence is used as an indicator of the quality of the experiment! In spite of the "non-analyticity the parameters of the dependence $\ln \bar{c}(z^{6/5})$ are included in empirical relations with rather precisely known parameters. The remaining part of the problem will be devoted to the study of diffusion of type B and the relations arising there.

Even though the linear dependence of $\ln \bar{c}(z^{6/5})$ cannot be analytically obtained, the parameters of this dependence are unambiguously expressed through the input parameters of the problem (for physical reasons - from dimensionless variables δ, D_{gb}/D and Dt/a^2, where a^3 is the volume of one grain of polycrystal). It can be written:

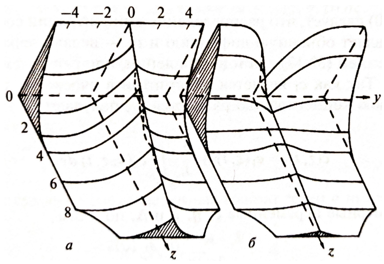


Figure 4. Diffusion concentration distributions in the B-type kinetic regime under different edge conditions (a – constant concentration at the boundary, b – constant number of particles in the system). The linear dependence of $\ln \bar{c}(z^{6/5})$ is observed in both cases.

$$\frac{\mathrm{d} \ln c}{\mathrm{d} z^{6/5}} = -C \delta^{\xi} a^{\psi} \left[\frac{D_{gb}}{D}\right]^{\eta} \left[\frac{Dt}{a^2}\right]^{\zeta}.$$

Here C is a numerical coefficient of order $\sim 10^0$.

C1	What can one say about the coefficients without resorting to numerical modeling? Find all values of coefficients that can be determined a priori.	**0.4**

C2	Find the remaining coefficients by numerical simulation. Take measurements in the range of $t \in [0.005; 0.200]$, $D_{gb}/D \in [5; 50]$, $\delta \in [0.02; 0.20]$.As a result of the program's operation, the Cavg.csv file contains the dependence of the logarithm of the average concentration $\ln \bar{c}$ of impurity particles on the penetration depth $z^{6/5}$, and in the file Cmap.csv – the dependence of the logarithm of the concentration $\ln c$ on the penetration depth $z^{6/5}$ and the y coordinate along the solid surface. These dependencies can be analyzed using the C.nb program.	**4.0**

C3	Find the value of C as accurately as possible.	**0.6**

Appendix №1

The software for the task is located in the ISPhO folder. The programs you will need are the following

- A.exe
- B.exe
- C.exe
- InteractiveBuffer.nb
- B.nb
- C.nb

The first three programs are used for modeling, the other three are used for visualization and processing of the obtained data. **Working with .exe** Files with .exe resolution provide a command line interface. When the program is started, it asks for a set of input parameters:

- A.exe is used in part **A**. The program asks for simulation time, crystal temperature and number of simulation repetitions. Time and temperature are measured in conventional units. The **activation energy must be given in the same units!** Increasing the number of repetitions and the simulation time increases the accuracy of the results but can be quite time-consuming. It is strongly recommended to run simulations with small values of these parameters first.
- B.exe is used in part **B**. The program asks for the simulation time, the relative diffusion coefficient of impurity at the polycrystal grain boundary and the relative volume of the boundary. Usually, the simulation is quite fast.
- C.exe is used in Part **C**. The program asks for the same set of parameters but takes a **bit** more time because it simulates the effect on a much smaller scale. It may be worth **planning** measurements in this part.

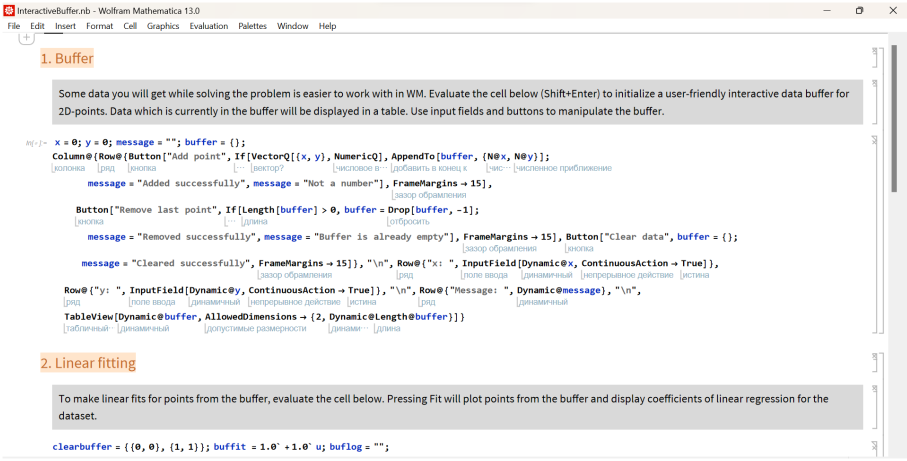
All programs display the expected simulation time. This may differ from the actual simulation time depending on the characteristics of your computer. Do not delete or edit the .csv files you create! Working with .nb To quickly and conveniently visualize and process results, programs with the .nb extension (so-called "*notebooks*") based on the Wolfram Mathematica computer algebra package are used.

- InteractiveBuffer.nb is a simple program that can memorize the points entered into it, graph them and draw a line through them by least squares method. The program exists as an "application tool"to solve the problem.
- B.nb is a tool that allows you to download actual simulation results from Part **B**, plot them as $\ln \bar{c}(z^2)$, and fit them to the requirements of the problem.
- C.nb is a tool that allows you to download actual simulation results from Part **C**, plot the $\ln \bar{c}(z^{6/5})$ dependence for them, and fit it according to the requirements of the problem. It also allows us to plot the detailed concentration distribution at the boundary of polycrystalline grains.

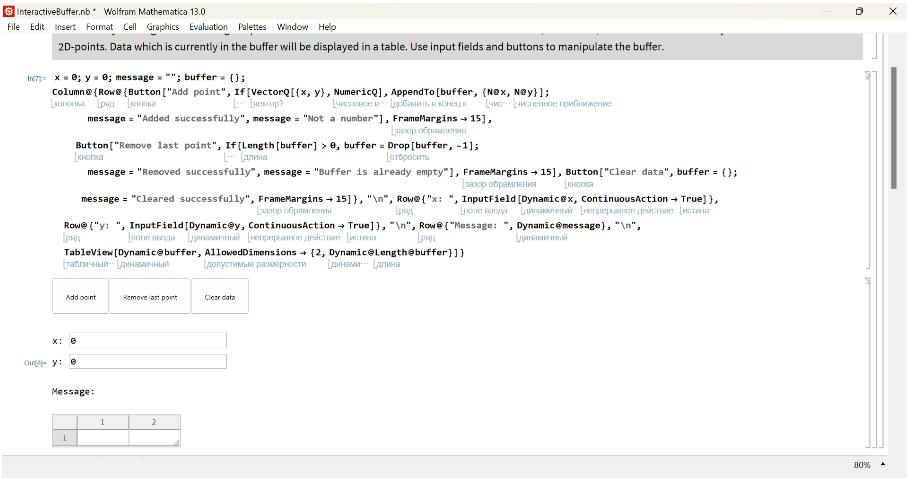
WARNING: Although it is technically possible to change the source code in .nb files, it is strongly discouraged! In case you break the source code, there will be no help in returning it to the initial state!

Appendix №2

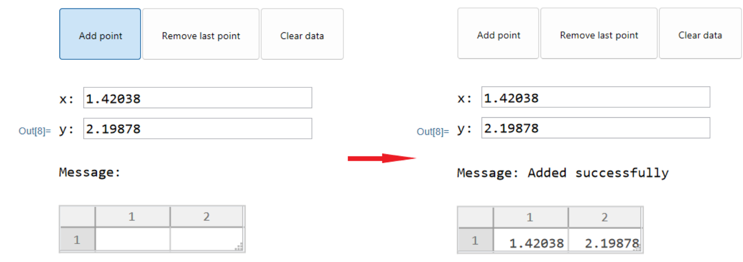
InteractiveBuffer.nb example To make working with Wolfram Mathematica easier for Olympiad participants, this appendix will provide a step-by-step guide to one of the programs, InteractiveBuffer.nb. After opening the file, its contents should appear on the screen (the details may look different depending on the WM version):



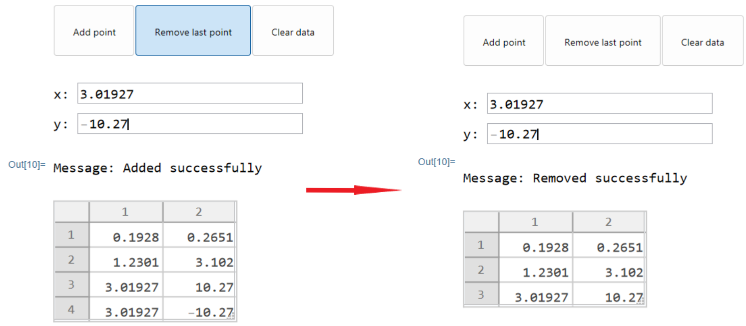
Click anywhere in the code cell and press Shift+Enter. This will execute the code in the cell. A new cell will appear at the bottom and will look like this:



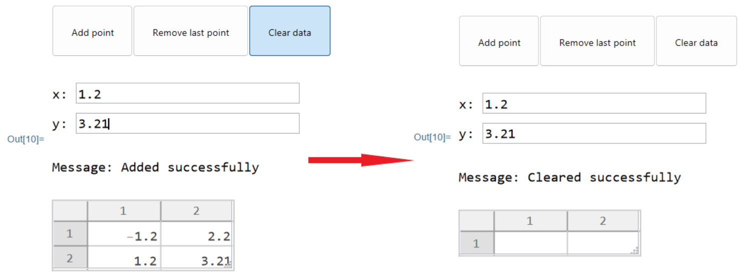
This is the interactive buffer that you can work with when solving the problem. You can enter x and y coordinates in the *interactive input fields*. If you then press the Add point button, these values are added to the buffer:



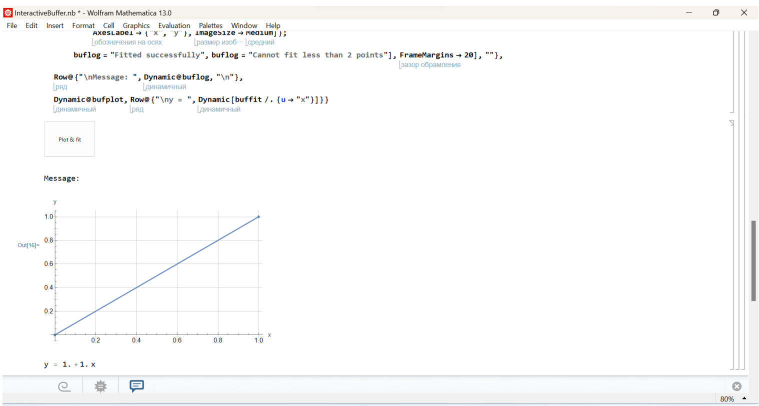
In case of adding an incorrect point to the buffer, the program provides the possibility to remove the last point in the buffer. To do this, click the Remove last point button:



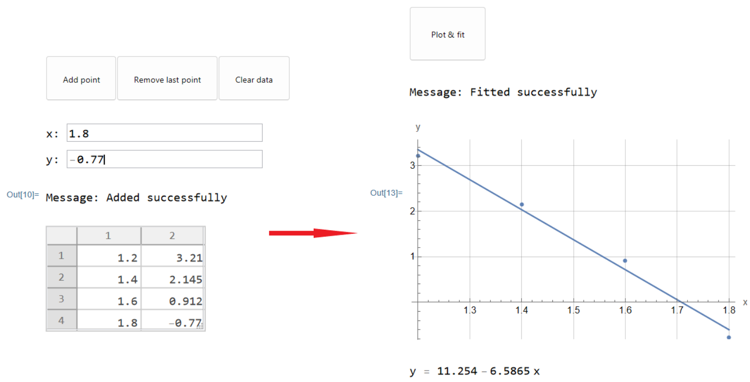
To completely clear the buffer, click Clear data:



After filling the buffer with the required points, evaluate the cell in the second section. The result will look as follows:



If you press the Plot fit button, the program will plot the points from the buffer and calculate linear regression coefficients for them. An example of the buffer contents and the corresponding fit result:



The rest of the .nb files look and work in a similar way, with comments and an intuitive interface.

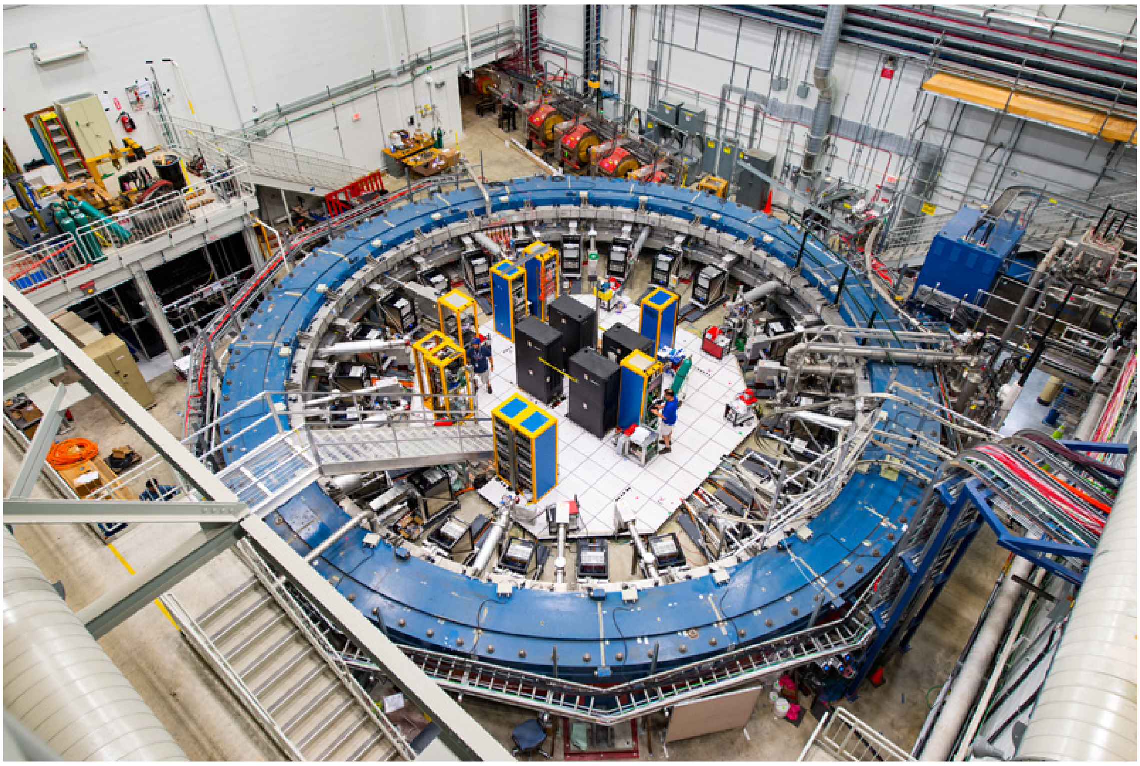
T2. Muon anomalous magnetic moment

Measurement of electron and muon magnetic moments is one of the most accurate in modern physics. This measurement provides a very precise test for elementary particles model. In the experiment positively charged muons (also called antimuons) are used. Antimuon mass is approximately 200 times larger than mass of the electron and antimuon charge is equal to the absolute value of electron charge. Throughout the problem we will refer to the positively charged muons as just muons for brevity.

Muon (and electron) has angular momentum $\vec{s}$, which is not connected with its motion in space. This angular momentum is called spin of the muon. Spin motion can be described by the torque equation, the rate of change of spin moment is equal to the applied torque. Moreover, muon has magnetic moment

$$\vec{\mu} = g_\mu \frac{q}{2m_\mu} \vec{s}.$$

Here m_μ and q are mass and charge of the muon, $g_\mu = 2(1 + a_\mu)$ is a dimensionless constant. The parameter a_μ describes the deviation of magnetic moment from «normal» value (which corresponds to $g_\mu = 2$). The problem deals with an experiment in which the anomalous contribution to magnetic moment a_μ is determined. Muons move in a circle in homogeneous magnetic field and their magnetic moment precession frequency is measured .



Storage ring for muon $g - 2$ measurement experiment, Fermilab

In particle physics energies are usually measured in electronvolts (eV). Instead of particle mass m the rest energy mc^2 is used, its value is also measured in electronvolts. You may use the following numerical values:

- 1 eV $= 1.602 \cdot 10^{-19}$ J
- Speed of light in vacuum $c = 2.998 \cdot 10^8$ m/s
- Muon mass $m_\mu c^2 = 105.7$ MeV
- Antimuon charge $q = 1.602 \cdot 10^{-19}$ C
- Muon is unstable, its lifetime in the rest frame is $\tau = 2.197 \cdot 10^{-6}$ s

Rotation can be described by an angular velocity vector $\vec{\omega}$, which is directed along the rotation axis, the direction of $\vec{\omega}$ is determined by the right-hand rule. If a vector $\vec{A}$ rotates in laboratory frame of reference with angular velocity $\vec{\omega}$, its time derivative is

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}.$$

Let the reference frame B (coordinates and time are x', y', z', t') move with respect to laboratory reference frame A (coordinates and time ar x, y, z, t) with velocity V along x axis. The coordinates in these frames are related by the Lorentz transformation

$$x = \gamma\left(x' + Vt'\right), \quad t = \gamma\left(t' + \frac{V}{c^2}x'\right), \quad y = y', \quad z = z', \quad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

In all your answers you can use the γ defined above to simplify formulas.

If particle's velocity in the moving reference frame B is $\vec{v}'$, with projections on coordinate axes are v'_x, v'_y, v'_z, its velocity components in the laboratory reference frame are

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}, \quad v_y = \frac{v'_y\sqrt{1 - V^2/c^2}}{1 + \frac{v'_x V}{c^2}}, \quad v_z = \frac{v'_z\sqrt{1 - V^2/c^2}}{1 + \frac{v'_x V}{c^2}}.$$

Energy and momentum of relativistic particle with mass m moving at speed $\vec{v}$ are

$$E = \gamma mc^2, \quad \vec{p} = \gamma m\vec{v}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

If particle moves under the force $\vec{F}$, the equation of motion is

$$\frac{d\vec{p}}{dt} = \vec{F}.$$

In questions that requier a numerical answer you need to calculate three significant figures.

Part A. Dynamics in magnetic field (4.5 points)

During anomalous magnetic moment measurement experiment a beam of muons moves in magnetic field $B = 1.450$ T, muons' momenta are perpendicular to the magnetic field and equal to $pc = 3.094$ GeV. The magnetic field is directed along the z axis.

A1 For a muon in the beam find the difference between speed of light and muons' speed $\Delta v = c - v$ (provide formula and compute numerical value). **1.0**

A2 For a muon in the beam find the kinetic energy E_k, that is the difference between the full energy and the rest energy (provide formula and compute numerical value in Gev). **0.5**

A3 Find radius R of muon's orbit and period T of its motion. (Provide formulas and compute numerical values.) **1.0**

A4 Due to instability a muon can orbit the circle only finite time. Find number of revolutions N the muon makes before its decay. **0.8**

Now we consider the case when muon is at rest in the same magnetic field B as in previous parts.

A5 Obtain a formula for the spin time derivative, $d\vec{s}/dt$, express the answer in terms of $\vec{B}$, $\vec{s}$, g_μ and fundamental constants. **0.7**

A6 Find the z component of angular velocity ω_z of muon's spin. Express the answer in terms of B, g_μ and fundamental constants. **0.5**

Part B. Thomas precession (3.5 points)

The muon spin is not retated to its spatial motion and should be described in the frame of reference in which the muon is at rest (such a frame is called comoving frame). If the muon is moving with acceleration, at each instant of time we must choose a different comoving frame of reference. Consider two successive Lorentz transformations. The first transformation is from the laboratory frame A (coordinates x, y, t) to the moving frame B (coordinates x', y', t'). The second is from the reference frame B to the third reference frame C (coordinates x'', y'', t''). It turns out that combination of this transformation is not a Lorentz transformation with some resulting velocity. There is an additional rotation which leads to the spin precession. This effect is called Thomas precession. Thomas precession is purely kinematics phenomenon, so to describe it we do not have to consider magnetic moment dynamics.

Let the reference frame B move with velocity v_x along x axis with respect to A. In turn reference frame C moves with velocity v'_y along y axis with respect to B. You may assume that $v'_y \ll v_x$ and work up to the first order in v'_y. To simplify your expressions you may use notation $\gamma = 1/\sqrt{1 - v_x^2/c^2}$.

B1 Find the x and y components of velocity $\vec{v}_C$ of reference frame C with respect to the reference frame A. Express the answer in terms of v_x, v'_y. **0.5**

B2 Find the x'' and y'' components of velocity $\vec{v}''_A$ of the reference frame A with respect to the reference frame C. Take into account that v''_y is small. **0.7**

B3 It turns out that $\vec{v}''_A \neq -\vec{v}_C$. It can be attributed to the fact that axes of the reference frame C are rotated with respect to the axes of frame A. Therefore all vectors in frame C are also rotated with respect to the frame A. Calculate the angle of rotation $\Delta\theta$ defined as an angle between vectors $-\vec{v}''_A$ and $\vec{v}_C$. The angle is positive if the rotation from first vector to the second vector viewed from the z axis is counterclockwise. (Axes x, y, z form right-handed coordinate system.) Express your answer in terms of v_x, v'_y, $\gamma = 1/\sqrt{1 - v_x^2/c^2}$, c. **0.6**

B4 Let a particle move with velocity $\vec{v}$ along x axis with respect to the frame A. Particle's acceleration is directed along the y axis, its projection is a_y. Frame B is the comoving frame for the particle at the considered moment (i.e. the velocity of the particle in it is zero) and moves with the speed v along x axis. After time interval dt (measured in laboratory reference frame) particle velocity is $\vec{v} + d\vec{v}$, corresponding comoving reference frame is C. From the previous question it follows that the axes of comoving frame rotate. Using results of the previous quaestion, find z component ω_T (angular velocity of Thomas precession) of the comoving frame angular velocity (in this case rotation is around z axis). Express the answer in terms of v_x, a_y, γ. **1.0**

B5 Let the muon move in homogeneous magnetic field B directed along z axis. Muon velocity is perpendicular to the magnetic field. Calculate the z component of Thomes precession angular velocity. Express the answer in terms of B, q, m_μ, γ. **0.7**

Part C. Magnetic moment precession (2 points)

Let us consider muon moving in homogeneous magnetic field. Muon spin precession is determined by two factors: direct interaction with magnetic field and Thomas precession due to accelerated motion. The angular velocity of precession due to magnetic field can be calculated by the formula, obtained in the question A6 in the case of muon at rest. Далее везде будем считать, что спин мюона параллелен плоскости движения мюона.

C1 Find the expression for z component of muon spin precession ω_s angular velocity with respect to laboratory reference frame. Express the answer in terms of B, g_μ, q, m_μ, γ. **0.7**

C2 In practice it is more convenient to consider spin rotation with respect to the direction of muon momentum. Find the angular velocity ω_a of muon spin rotation with respect to the direction of its momentum (that is also equal to the time derivative of the angle between momentum and spin). Express the answer in terms of B, a_μ, q, m_μ, γ. **0.5**

C3 Experimentally measured frequency of muon spin precession with respect to the momentum direction is $f_a = 229081$ Hz. Find the expression for muon anomalous magnetic moment a_μ. Express the answer in terms of f_a, B, q, m_μ, γ. Compute the numerical value of a_μ. **0.8**

T3. Frequency measurements with ring resonators

This task deals with methods for high-precision frequency measurements. During the solution, you will learn two measurement methods: signal modulation (part C) and frequency comb (part D).

Why do you need more precision?

- The accuracy of physical measurements is important when we determine the values of fundamental constants. Many times in the history of physics, small discrepancies between experimental results and theory have required revision of the entire theoretical model and led to major discoveries in fundamental physics.
- High accuracy of frequency measurement is necessary for normal operation of satellite communication systems, navigation, ground-based telecommunication systems, etc.

In the framework of classical physics, any measurement is a comparison with a standard. As a frequency standard we will use the natural frequencies of a **ring optical resonator,** because today the best resonators are ring resonators. And the higher the goodness, the more accurate is the value of the reference natural frequency. In Part B, you will learn more about ring resonators with the example of **a ring fiber optical resonator**. In Part A, you will derive the theory of wave propagation in the optical fiber from which the resonator described in Part B is made. At the very end of the problem there are reference materials: Maxwell's equations and differential operators. If you know all the physical laws that are included in the Olympiad program, you will **not need reference materials to solve the problem**.

Equations of EM wave propagation The electromagnetic field at each point of space is described by four vectors: $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{D}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ — electric field strength, magnetic field strength, electric induction and magnetic induction. These vectors are related to the $\vec{P}$ and magnetisation $\vec{M}$ of the medium by realations:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P},$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}),$$

where ε_0 is the electric constant and μ_0 is the magnetic constant.

Use following approximations when solving the problem:

- All media are non-magnetic and $\vec{M} = 0$.
- There are no free charges.
- The response of the medium is instantaneous and local, that is, $\vec{P}(\vec{r},t)$ depends only on $\vec{E}(\vec{r},t)$.

EM wave theoey. By applying the rot operator to Maxwell's equations, Faraday's induction law in differential form and the theorem on the circulation of magnetic field strength in differential form (see References), we obtain the basic equations of electromagnetic wave propagation:

$$\Delta\vec{E} = \mu_0\frac{\partial^2\vec{D}}{\partial t^2}.$$

$$\Delta\vec{H} = \varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}.$$

Part A. Field profile in fiber optics (2.5 points)

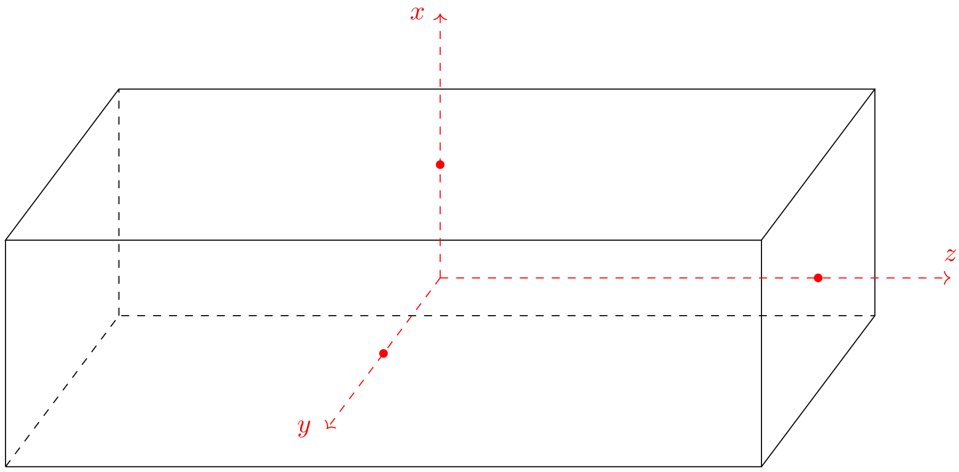
In parts $\mathrm{A}, \mathrm{B}, \mathrm{C}$, we will assume that the polarization depends linearly on the electric field strength: $\vec{P} = \varepsilon_0(\varepsilon(\omega)-1)\vec{E}$, where $\varepsilon(\omega)$ is the dielectric permittivity of the medium, which usually depends on the frequency of the electromagnetic wave in the medium. Dielectric permittivity of vacuum is equal to 1.

So, the equations of propagation of electromagnetic waves can be written in this form:

$$\Delta\vec{E} = \mu_0\varepsilon\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}.$$

$$\Delta\vec{H} = \mu_0\varepsilon\varepsilon_0\frac{\partial^2\vec{H}}{\partial t^2}.$$

International Scientific Physics Olympiad



A plane monochromatic electromagnetic wave of frequency ω propagates along a dielectric waveguide along the z-axis (fig. 1), which is a rectangular parallelepiped of material with dielectric constant $\varepsilon(\omega)$ with sides a, L_1 и L_2. $L_1, L_2 \to \infty$ and are much more than any characteristic lengths in this problem.

Let's introduce the Cartesian coordinate system $Oxyz$. O ccoincides with the center of the parallelepiped. The edges of length L_1 are parallel to z-axis, length a – to x-axis, length L_2 – to y-axis. $(\hat{x}, \hat{y}, \hat{z})$ is a right triple.

Consider a wave polarized parallel to the y-axis: $\vec{E}||\hat{y}$. We will look for the solution for the field $\vec{E}$ in the waveguide in the following form:

$$\vec{E}(\vec{r}, t) = E_0 \operatorname{Re}\Big(A(x, y)\exp(i\omega t - i\beta z)\Big)\hat{y},$$

where E_0 has the dimensionality of the electric field, and $A(x, y)$ is a dimensionless complex quantity that contains information about the ratio of amplitudes and phase differences of the field at different points of the waveguide. Without restriction of generality $A(0, 0) = 1$.

Let us also consider that $L_2 \gg a$, so it will be assumed that $\partial A/\partial y \ll \partial A/\partial x$, and hence A is only dependent on x.

The solution for $A(x)$ can be sought in the form:

$$A(x) = \begin{cases} \cos(k_{in}x) \quad \text{при} \quad |x| < a/2 \\ C_1 \exp(k_{out}x) + B\exp(-k_{out}x) \quad \text{при} \quad x > a/2 \\ B\exp(k_{out}x) + C_2\exp(-k_{out}x) \quad \text{при} \quad x < -a/2 \end{cases}$$

where $k_{in} > 0, k_{out} > 0$.

A1	What are the coefficients C_1 and C_2?	**0.3**

A2	Express k_{in} and k_{out} in terms of $\varepsilon, \omega, \beta$ and speed of light in vacuum $c = 1/\sqrt{\varepsilon_0\mu_0}$, using the wave equation.	**0.5**

To find B, we need to write down the boundary conditions for the boundary $x = a/2$ (for the boundary $x = -a/2$ we get the same equations because $A(x)$ is an even function).

A3	Prove that the tangential components of the electric field strength are equal on both sides of the interface (as in electrostatics).	**0.3**

A4	Write the boundary conditions for the tangential component of the electric field. Express B in terms of a, k_{in}, k_{out}.	**0.3**

Now we need to derive an equation for finding $\beta(\omega)$. For this purpose we can write another boundary condition – for $\vec{H}$.

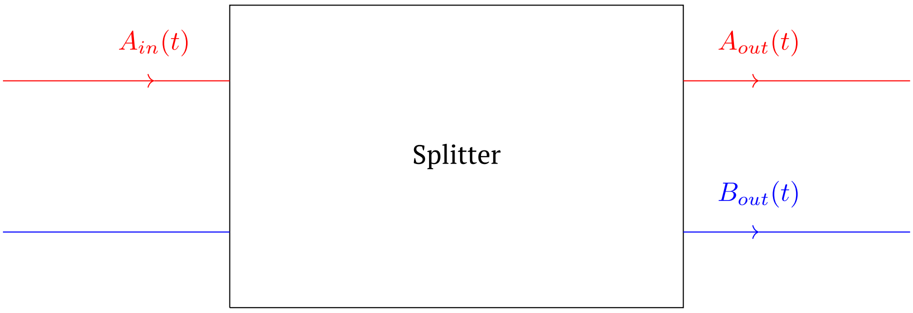
A5	Prove that the tangential components of the magnetic field strength are equal on both sides of the interface (as in magnetostatics).	**0.3**

A6	Prove that $$H_z = E_0 \,\mathrm{Re}\left(\frac{i}{\mu_0\omega}\frac{\partial A}{\partial x}\exp(i(\omega t-\beta z))\right).$$	**0.3**

A7	Write the boundary conditions for the tangential component of the magnetic field $\vec{H}$. The answer can be expressed in terms of k_{in}, k_{out}, a.	**0.3**

A8	Substitute the values k_{in}, k_{out}, obtained in A2 into the equation from A7. Obtain an equation from which βcan be determined (this equation is solved numerically only). The equation can include $\beta, \omega, \varepsilon, c, a$.	**0.2**

Part B. Fiber resonator transmittance (3.0 points)

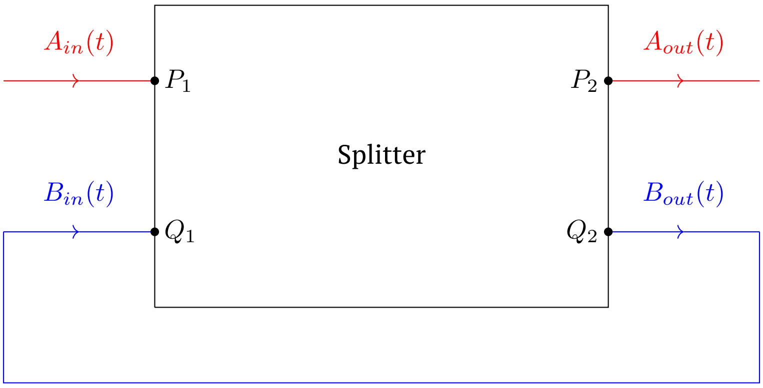


The radiation from the external laser enters a quadrupole beam splitter, which is an optical system with two pairs of coupled channels: (P_1, P_2) and (Q_1, Q_2).

A monochromatic wave with complex amplitude $A_{in}(t)$ entering the splitter is split into two outgoing waves of smaller amplitude propagating in the same direction. The first of them propagates along the same pair channel as the incoming wave, and its complex amplitude is $A_{out}(t) = -r_s A_{in}(t)$. The second wave propagates through the channel of the other pair and its amplitude is $B_{out}(t) = it_s A_{in}(t)$. **Waves incident on Q_1 are divided similarly for those incident on P_1.** Consider that r_s and t_s are real positive numbers and $t_s \ll 1$.

B1	Assuming that there is no energy loss in the divider, find the relationship between r_s and t_s.	**0.2**

A fiber resonator can be assembled with the divider. In order to do so, it is necessary to connect the channels Q_1 and Q_2 with a loop of single-mode fiber (the equation for $\beta(\omega)$ has the only solution for the frequencies considered in this part) of length L. The time for light of frequency ω to travel through the loop is $\tau(\omega)$. Also note that due to material absorption, the amplitude of the wave in the loop decreases by a factor $1/\kappa$ for each round in the loop $(1-\kappa \ll 1)$. For all tasks in parts B and C **neglect backscattering**.



Let a monochromatic wave $A_{in}(t) = A_0 e^{i\omega t}$ from an external laser be incident on the input of channel P_1.

B2	Express the field amplitude at the input to Q_1 of the splitter $B_{in}(t)$ in terms of κ and the field amplitude at the output of Q_2 at time $(t-\tau(\omega))$ — $B_{out}(t-\tau(\omega))$.	**0.3**

Assume the system has come to the steady-state regime, i.e. the absolute values of complex field amplitudes at any point are constant, and the phase difference of the field at two points does not change with time. Then for any amplitude E_i (instead of E_i we can substitute $A_{in}, A_{out}, B_{in}, B_{out}$) the expressions are valid:

$$E_i(t - t') = e^{-i\omega t'} E_i(t).$$

B3	Using the stationarity conditions, express $B_{in}(t)$ in terms of $B_{out}(t), \kappa, \omega, \tau$.	**0.3**

B4	Express $B_{in}(t)$ in terms of $A_0, r_s, t_s, \kappa, \omega, \tau$ and t, using the result of the previous task.	**0.5**

B5	What is the power N_2, leaving the channel P_2? Express the answer in terms of $\omega\tau(\omega), \kappa, r_s$ and the power N_1 in channel P_1.	**0.5**

B6 Sketch a qualitative plot of $N_2/N_1(\omega\tau)$ for fiber resonator with the following parameters: **0.6**

- $\kappa = 1 - 5 \cdot 10^{-3}$;
- $t_s = 0.1$.

At what values of $\omega\tau$ is the ratio N_2/N_1 minimal?

B7	Find the sharpness Q of the absorption peak with number $n = 100$. *Sharpness is the ratio of the peak frequency to the width of the region of frequencies for which the transmission dip is not less than half of the maximum dip of a particular peak.*	**0.6**

Today it is possible to create a fiber resonator based on a beam splitter with peak sharpness of about $5 \cdot 10^7$ near the frequency corresponding to the wavelength of 1550 nm. And if we use a ring microresonator made of fused quartz, we can obtain a sharpness of 10^9. Resonators with such small losses open us a wide range of possibilities for accurate frequency measurements.

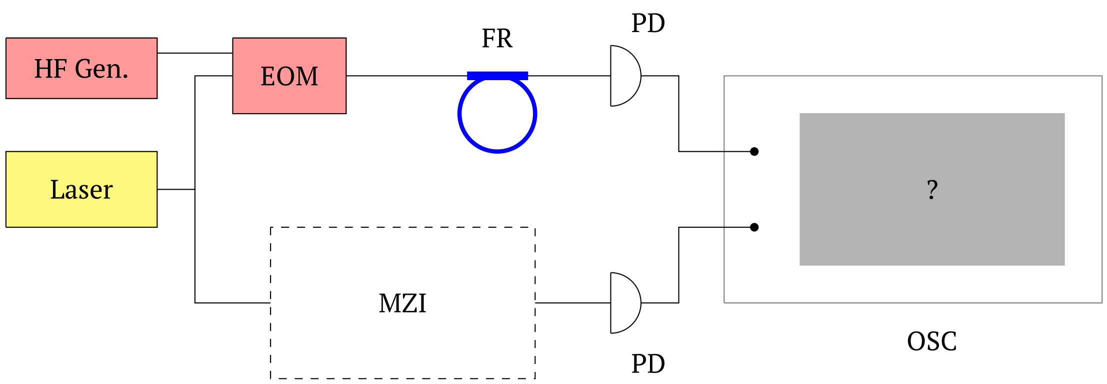
That is, with a quartz microresonator we can obtain a spectral line with an average wavelength of 1550 nm with a width of only 0.000002 nm!

Part C. Calibration of the Mach-Zehnder Interferometer (2.0 points)

Consider the setup for calibrating the interferometer shown in Fig.1 Now the laser frequency varies smoothly with time (the derivative $\omega\tau$ is much smaller than the inverse time for establishing the stationary mode in the resonator), and its power is constant.

$$\omega(t) = \omega_0 + \alpha t$$

The radiation flux from the laser is split into two arms by another beam splitter similar to the splitter in part B. The first arm of the optical circuit contains a Mach-Zehnder calibrated interferometer and a photodetector that measures the power coming out of it, and the second arm contains a resonator and a similar photodetector. The signals from the photodetectors are fed to an oscilloscope. The signal of the photodetector is proportional to the radiation power. **The oscilloscope shows the time dependence of the photodetector voltages (not XY mode!).**



Laser parameters:

- $\omega_0 = 1.26 \cdot 10^{15}$ c^{-1}
- $\alpha = 5 \cdot 10^9$ c^{-2}

Resonator parameters:

- Sharpness of absorption peaks near ω_0 is $5 \cdot 10^7$
- Consider that $\tau(\omega)$ independent of frequency and equal to $\tau_0 = 3 \cdot 10^{-11}$ c
- $\kappa = 1 - 1 \cdot 10^{-3}$
- $t_s = 0.1$

Parameters of the interferometer to be calibrated:

- Its **power** throughput capacity is approx

$$F_{MZI}(\omega) = \cos^2(\omega/2\omega_{MZI})$$

- $\omega_{MZI} \approx 1250MHz$. The exact value will be determined at this setup.

C1	Sketch a qualitative plot of the oscilloscope readings if it is known for sure that the frequency of the tunable laser reaches exactly one of the frequencies given in B6 (these are the frequencies where the FR transmittance is minimal).	**0.5**

Now an amplitude electro-optic modulator is connected to the second arm. Its **amplitude** bandwidth is independent of the radiation wavelength and is equal to:

$$f_{EOM}(t) = \beta + (1-\beta)\cos(\Omega t)$$

$$\Omega \ll \omega_0$$

$$\beta < 1$$

C2	Draw what the oscilloscope will show when $\Omega \approx 220MHz$. Note that $\alpha \ll \Omega\omega_0$.	**1.0**

C3	Estimate with what relative accuracy can the period ω_{MZI} be measured on this setup if the maximum signal frequency of the high-frequency oscillator is $\Omega_{max} = 1250MHz$?	**0.5**

Part D. Cubic nonlinearity and solitons (2.5 points)

For precision measurements and high-frequency communication, it is necessary to obtain very short light pulses that do not change their profile with time (such pulses are called solitons). This is not possible in linear optics because the dielectric constant and the dispersion constant depend on frequency, and the pulses do "blur"due to dispersion.

Fortunately, the approximations of linear optics do not work for all materials, and the resulting nonlinearities can compensate for the undesirable effects associated with dispersion.

In Part D, we will assume that the projection $P_i(\vec{r},t)$ of the polarisation vector on an arbitrary axis i depends on the projection of the electric field strength $E_i(\vec{r},t)$ at the same point at that instant of time as follows:

$$P_i(\vec{r},t) = \varepsilon_0((\varepsilon(\omega)-1)E_i(\vec{r},t) + \chi E_i^3(\vec{r},t))$$

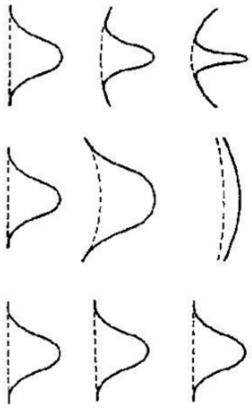
The pulse is a sum of electromagnetic waves with frequencies very close to the carrier frequency ($\omega \approx \omega_0$), and propagation constants very close to the propagation constant of the carrier wave $\beta \approx \beta_0 = \beta(\omega_0)$:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left(E_0 F(z,t) A(x,y)\exp(i\omega_0 t - i\beta_0 z)\right)\hat{y},$$

Here $\beta(\omega)$ is the spreading constant found in part A, $A(x,y)$ is also taken from part A, $F(z,t)$ is a complex function that contains information about the momentum profile.

The function $\beta(\omega)$ can be approximated by a quadratic function:

$$\beta = \beta_0 + \beta_1(\omega-\omega_0) + \beta_2(\omega-\omega_0)^2/2$$



We can substitute this type of solution into the wave equation, which was derived in the introduction, and by mathematical transformations beyond the school program, obtain the equation for $F(z,t)$;

$$\frac{\partial F}{\partial z}+\beta_1\frac{\partial F}{\partial t}+\frac{i\beta_2}{2}\frac{\partial^2 F}{\partial t^2}-i\gamma|F|^2F=0$$

The pulse will propagate with group velocity, so it is reasonable to move to a frame of reference that will also move with group velocity. In this system the last equation will be rewritten in the form:

$$\frac{\partial F}{\partial z}+\frac{i\beta_2}{2\beta_1^2}\frac{\partial^2 F}{\partial s^2}=i\gamma|F|^2F,$$

where $s=t/\beta_1-z$ – is the spatial coordinate in the soliton reference frame, and γ is the cubic nonlinearity coefficient, which is proportional to χ (и имеет тот же знак). The resulting equation is called the **Nonlinear Schrödinger equation** (hereafter referred to as the NLSE). In the Part D assignment, you will **analyze the NLSE**. You will need to express all answers through its coefficients.

D1	Express the group velocity v_g through β_1.	**0.1**

D2 Let us find the form of the soliton. The solution of NLSE can be found in the form: **0.6**

$$F(z,s)=\frac{F_0\exp(i\sigma z)}{\cosh(\theta s)}.$$

Experess F_0 и σ in terms of θ,β_1,β_2 и γ.

Let one of the natural frequencies of FR be ω_0. The length of FR is such that there are many other natural frequencies in the range from 0 to ω_0 находится очень много других собственных частот. Then the following equality is true for the other natural frequencies: $\omega_\mu\approx\omega_0+D_1\mu+D_2\mu^2/2$, where μ is integer and $\omega_0\gg D_1\gg D_2$.

D3	Express D_1 and D_2 in terms of β_1,β_2 and loop length L. *Hint: the BP natural frequency criterion:* $B_{in}(t)$ *and* $B_{in}(t-\tau(\omega_\mu))$ *have the same phase.*	**0.5**
D4	Let the resonator be made of a material with $\chi>0$. At what D_2 can solitons exist in it?	**0.3**
D5	Let a soliton with carrier frequency ω_0. circulates in the FR described in D3. The external laser does not work. Plot the emission spectrum of the resonator (the dependence of specific power on frequency) **qualitatively** in the frequency range $(\omega_0-20D_1,\omega_0+20D_1)$. Consider that $\omega_0/Q(\omega_0)\ll D_1$. (Remember that Q -is sharpness defined in B7)	**0.6**
D6	Estimate the absolute error of the angular frequency measurement ω using the spectrum from item D5.	**0.2**
D7	Express the round-trip time τ_s through D_1.	**0.2**

Reference materials

Let us introduce a Cartesian coordinate system $Oxyz$. in space. The unit vectors along the corresponding axes are $\widehat{x}, \widehat{y}, \widehat{z}$ and form the right triple.

- The grad operator converts a scalar φ into a vector as follows:

$$\mathrm{grad}\,\varphi = \widehat{x}\frac{\partial\varphi}{\partial x} + \widehat{y}\frac{\partial\varphi}{\partial y} + \widehat{z}\frac{\partial\varphi}{\partial z}$$

- The div operator converts vector $\vec{a} = a_x\widehat{x} + a_y\widehat{y} + a_z\widehat{z}$ into a scalar as follows:

$$\mathrm{div}\,\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

- The rot operator converts vector $\vec{a} = a_x\widehat{x} + a_y\widehat{y} + a_z\widehat{z}$ into a vector as follows:

$$\mathrm{rot}\,\vec{a} = \widehat{x}\Big(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\Big) + \widehat{y}\Big(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\Big) + \widehat{z}\Big(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\Big)$$

- The Laplace operator Δ converts a scalar to a scalar or a vector to a vector:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\Delta\vec{a} = \frac{\partial^2\vec{a}}{\partial x^2} + \frac{\partial^2\vec{a}}{\partial y^2} + \frac{\partial^2\vec{a}}{\partial z^2}$$

- If we apply rot twice to the vector $\vec{a}$ we get the vector:

$$\mathrm{rot}\,\mathrm{rot}\,\vec{a} = \mathrm{grad}\,\mathrm{div}\,\vec{a} - \Delta\vec{a}.$$

Maxwell's equations

Each equation is written first in integral form and then in differential form:

- Gauss's theorem for magnetic field induction (the flux of magnetic induction through any closed surface is 0):

$$\oint_S (\vec{B}\cdot d\vec{S}) = 0$$

$$\mathrm{div}\,\vec{B} = 0$$

- Gauss' theorem for electric induction (the flux of electric induction through a closed surface is equal to the free charge inside the volume bounded by it):

$$\oint_S (\vec{D}\cdot d\vec{S}) = q_f = \oint_V \rho_f dV$$

$$\mathrm{div}\,\vec{D} = \rho_f$$

here ρ_f is the bulk density of free charges.

- Faraday's law of electromagnetic induction (the rate of change of the flux of magnetic induction through an unclosed surface, taken with the opposite sign, is equal to the circulation of the electric field on a closed loop, which is the boundary of the surface):

$$\oint_l (\vec{E}\cdot d\vec{l}) = -\frac{\partial}{\partial t}\oint_S (\vec{B}\cdot d\vec{S})$$

$$\mathrm{rot}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

- Circulation theorem for the magnetic field strength (the sum of the rate of change of the electric induction flux through an unclosed surface and the total electric current of free charges through it is equal to the magnetic field circulation on a closed contour, which is the boundary of the surface):

$$\oint_l (\vec{H} \cdot d\vec{l}) = I_f + \frac{\partial}{\partial t} \oint_S (\vec{D} \cdot d\vec{S})$$

$$\mathrm{rot}\, \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_f$$