International Scientific Physics Olympiad

T1. Polycrystal diffusion

A1 Write the expression for v in terms of T and physical constants. From this, obtain the law of proportionality $D(T)$.

From thermodynamics we know that $v = \sqrt{3kT/m}$, where m is the particle's mass, and k is Boltzmann constant. Because characteristic jump time between inter-nodes is inversely proportional to transition probability × particle's speed, and because diffusion coefficient is inversely proportional to jump time, then $D(T) \propto v\exp(-E_g/kT) \propto \sqrt{T}\exp(-E_g/kT)$.

A2 Write down how the rms displacement of a particle Δr_{sq}, diffusion coefficient D and simulation time t are related to each other in the **three-dimensional case**. *Note:* the particle motion along each of the three axes in random walk can be considered independent. The diffusion coefficient D in **one-dimensional case is** related to the cell size a and characteristic jump time τ as $D = a^2/2\tau$. *Note:* if you cannot complete this task, in the future, consider $\Delta r_{sq} = \sqrt{Dt}$.

First, let's find rms particle displacement along one of the axes (we'll call this axis x). In this case movement will happen analogously to the one considered in preparational lectures, so rms particle displacement will be:

$$\overline{x^2} = na^2,$$

where n is total numner of jumps. Number of jumps can be estimated as:

$$n = t/\tau.$$

Considering $D = a^2/2\tau$, we obtain:

$$\overline{x^2} = 2Dt.$$

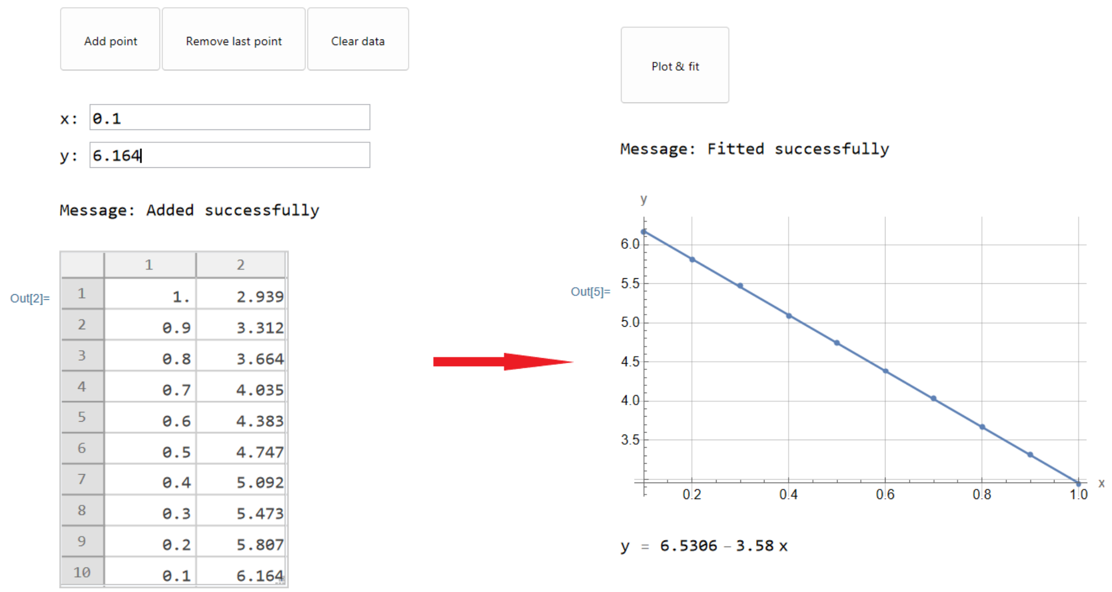
Finally, because movement along different axes can be considered as independent, then $\overline{y^2} = \overline{z^2} = 2Dt$. Using Pythagorean theorem, we obtain:

$$\Delta r_{sq} = \sqrt{\overline{x^2} + \overline{y^2} + \overline{z^2}} = \sqrt{6Dt}.$$

A3 Get the points to find $D(T)$ and linearize the resulting dependence. Find the energy E_g in arbitrary units with an accuracy of at least 2%. Explicitly state how you achieve this accuracy. Make measurements in $T \in [1;10]$, $t \in [1;10]$, and $N \in [10^2;10^4]$.

Let's use results of the previous task to find diffusion coefficient. Because $\Delta r_{sq} = 6Dt$, then $D = \Delta r_{sq}^2/6t$. Combining this with results of task **A1**, we get $\Delta r_{sq}^2/6t\sqrt{T} = \exp(-E_g/kT)$. The easiest way now is to find the slope of dependency of $Y = \ln\left[\Delta r_{sq}^2 t^{-1} T^{-1/2}\right]$ on $X = 1/T$. How the accurace is guaranteed: First, it'd be good to have $\Delta r_{sq} \gg 1$. Second, $1/T$ changes by 0.9, and E_g is proportional to the dependency slope. To guarantee a 2% accuracy we need the sum of errors for min and max temperatures to be $\sigma_y(0.1) + \sigma_y(1.0) \leq 0.02 \cdot 0.9E_g$. Because $y = \ln[\Delta r_{sq}^2 T^{-1/2}]$, then $\varepsilon_r(0.1) + \varepsilon_r(1.0) \leq 0.009E_g$. We can simply check that for all points $\varepsilon_r \leq 0.0045E_g$. Let's obtain data points and put them in a table below:

t	10	10	10	10	10	10	10	10	10	10
T	1.000	1.111	1.250	1.429	1.667	2.000	2.500	3.333	5.000	10.00
N	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000
Δr_{sq}	13.750	17.009	20.882	25.999	32.154	40.362	50.733	65.932	86.227	122.632
σ_r	0.053	0.066	0.081	0.102	0.127	0.156	0.198	0.257	0.336	0.477
X	1.000	0.900	0.800	0.700	0.6	0.5	0.4	0.3	0.2	0.1
Y	2.939	3.312	3.664	4.035	4.383	4.747	5.092	5.473	5.807	6.164



One can see that the required accuracy is indeed achieved. Let's plot $Y(X)$ and find its slope using InteractiveBuffer.nb. It will be equal to

$$-E_g = -3.58 \pm 0.02.$$

B1 What will be the coefficient α in such a structure?
Note: It is not necessary to solve this task to proceed to numerical modeling.

From the question paper we can conclude that grain boundaries which are perpendicular to the diffusion flow do not affect the effective diffudion coefficient, and then it will be:

$$D_{eff} \approx 2\delta D_{gb}/3 \implies \alpha = 2/3.$$

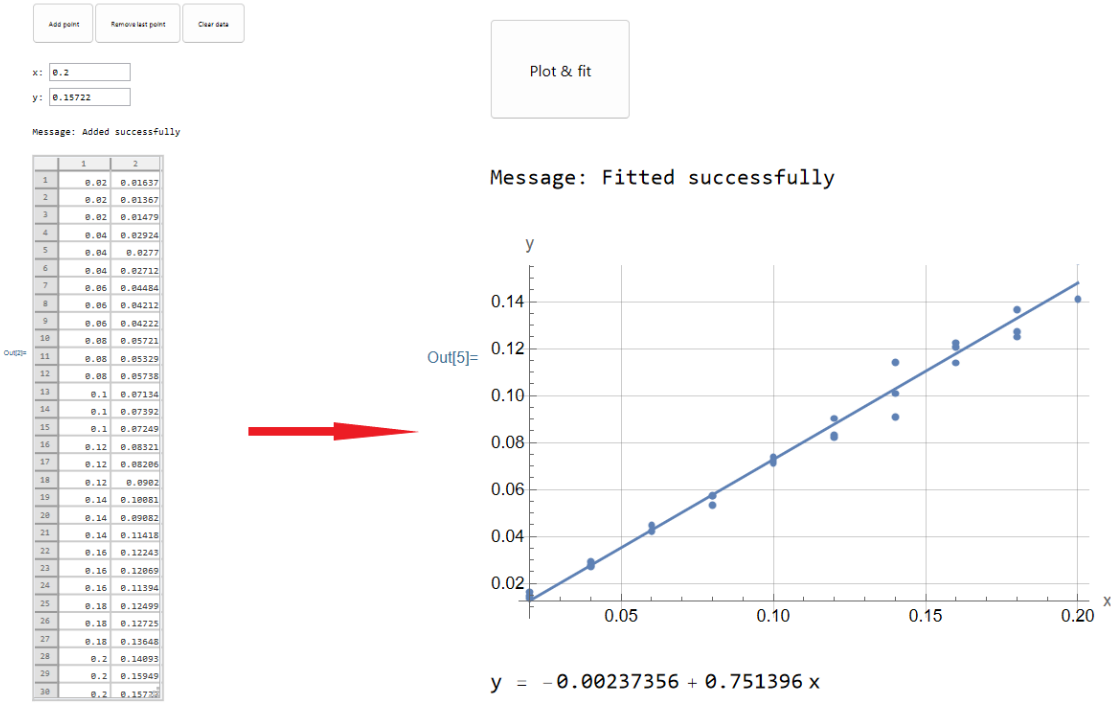
B2 Simulate diffusion in a polycrystal with $t \in [2;5]$, $D_{gb}/D \in [2;5]$, and $\delta \in [0.005;0.05]$, to obtain the experimental value of α. Compare with the result of **B1**.
Note: As a result of the program operation, the dependence of the logarithm of the mean concentration $\ln \bar{c}$ of impurity particles on the square of the penetration depth z^2 is written to the file Bavg.csv. This dependence can be analyzed with the program B.nb.

It is clear that simulation results will be better for larger D_{gb}/D and smaller δ. It is also better to use maximum simulation time, since type A diffusion appears on large timescales. So, let's choose $(5, 5, 0.005)$ as our starting point. Plots in B.nb should ideally have a slope of $k = -\frac{1}{4D_{eff}t}$, so in arbitrary units we write:

$$D_{eff} = \frac{1}{4|k|t} = 1 + \alpha\delta(D_{gb}/D - 1).$$

Choosing $X = \delta(D_{gb}/D - 1)$ and $Y = \left[\frac{1}{4|k|t} - 1\right]$ as coordinates makes this dependency linear. The best way to make X change considerably is to increase δ. We can also suggest lowering Width slider in B.nb to mitigate possible edge effects. Let's obtain data points, put them in a table below and make a plot in InteractiveBuffer.nb:

t	5	5	5	5	5	5	5	5	5	5
D_{gb}/D	5	5	5	5	5	5	5	5	5	5
δ	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040	0.045	0.050
$\lvert k\rvert$	0.049194 0.049325 0.049271	0.048579 0.048652 0.048679	0.047854 0.047978 0.047974	0.047294 0.047470 0.047286	0.046670 0.046558 0.046616	0.046159 0.046208 0.045863	0.045421 0.045837 0.044876	0.044546 0.044615 0.044885	0.044445 0.044355 0.043995	0.043823 0.043122 0.043207
Y	0.01637 0.01367 0.01479	0.02924 0.02770 0.02712	0.04484 0.04212 0.04222	0.05721 0.05329 0.05738	0.07134 0.07392 0.07259	0.08321 0.08206 0.09020	0.10081 0.09082 0.11418	0.12243 0.12069 0.11394	0.12499 0.12725 0.13648	0.14093 0.15949 0.15722
X	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20



	1	2
1	0.02	0.01637
2	0.02	0.01367
3	0.02	0.01479
4	0.04	0.02924
5	0.04	0.0277
6	0.04	0.02712
7	0.06	0.04484
8	0.06	0.04212
9	0.06	0.04222
10	0.08	0.05721
11	0.08	0.05329
12	0.08	0.05738
13	0.1	0.07134
14	0.1	0.07392
15	0.1	0.07249
16	0.12	0.08321
17	0.12	0.08206
18	0.12	0.0902
19	0.14	0.10081
20	0.14	0.09082
21	0.14	0.11418
22	0.16	0.12243
23	0.16	0.12069
24	0.16	0.11394
25	0.18	0.12499
26	0.18	0.12725
27	0.18	0.13648
28	0.2	0.14093
29	0.2	0.15949
30	0.2	0.157[illegible]

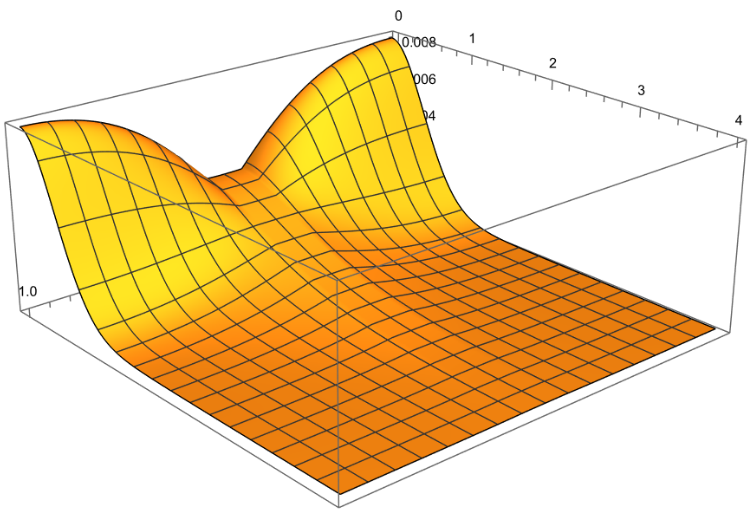
Simulation result give us $\alpha = 0.75 \pm 0.09$ which coincides with theoretical estimation within a margin of error.

C1 What can one say about the coefficients without resorting to numerical modeling? Find all values of coefficients that can be determined a priori.

The only "a priori"constraint here is the dimentional one. Conveniently, every factor in the r.h.s. except for a is dimentionless, so $\psi = -6/5$.

C2 Find the remaining coefficients by numerical simulation. Take measurements in the range of $t \in [0.005; 0.200]$, $D_{gb}/D \in [5; 50]$, $\delta \in [0.02; 0.20]$.As a result of the program's operation, the Cavg.csv file contains the dependence of the logarithm of the average concentration $\ln \bar{c}$ of impurity particles on the penetration depth $z^{6/5}$, and in the file Cmap.csv – the dependence of the logarithm of the concentration $\ln c$ on the penetration depth $z^{6/5}$ and the y coordinate along the solid surface. These dependencies can be analyzed using the C.nb program.

As it was stated in the question paper, the scale of type B diffusion should be much less than a (i.e. $\sqrt{2[D_{gb}/D]t} \ll 1$) and simultaneously much more than δa (i.e. $\sqrt{2t} \gg \delta$). Then, if we for example set input paramenters to be $(0.100; 25; 0.10)$, we will be able to achieve type B diffusion by varying any of input parameters in a wide range. We always can check whether simulated diffusion was of the type B or not by plotting the concentration profile to see if it looks like the one in the question paper. For example, here is concentration map for initial parameters $(0.100; 25; 0.10)$:



To find the slope of $k = \dfrac{\mathrm{d}\ln\bar{c}}{\mathrm{d}z^{6/5}}$ we will clip linear regions of plots generated in C.nb.

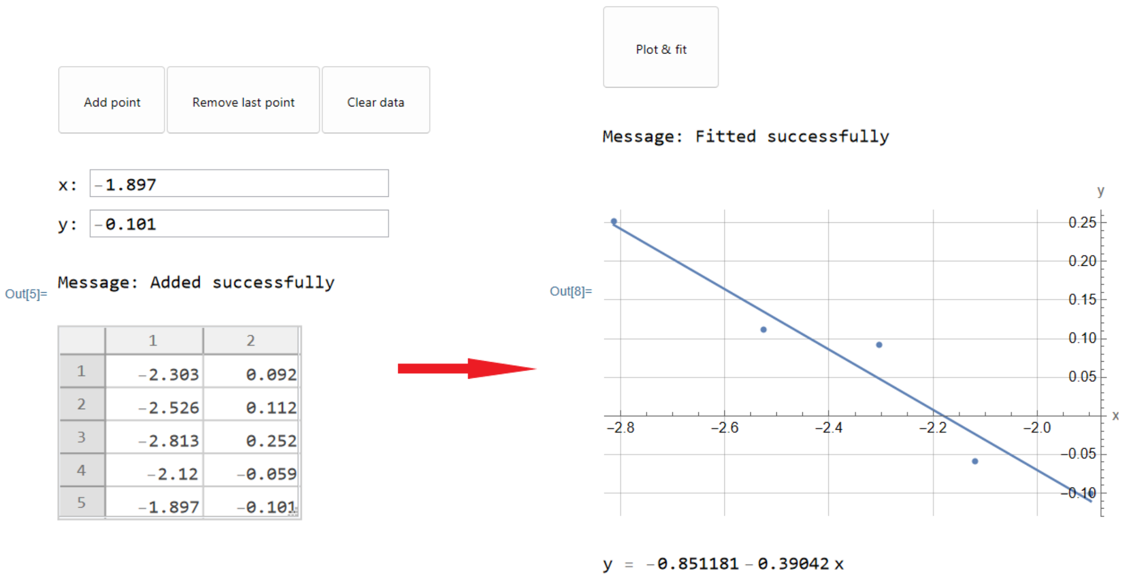
We can obtain ξ by simulating diffusion with different δ and plotting $\ln|k|(\ln\delta)$ in InteractiveBuffer.nb. η and ζ can be found in a similar manner (varying $[D_{gb}/D]$ and t correspondingly).

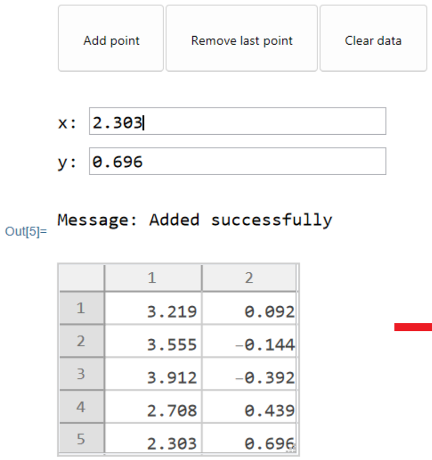
Let's obtain data points and put them in a table below ($\delta^{-0.39}[D_{gb}/D]^{-0.68}t^{-0.44} \equiv Z$):

δ	0.100	0.080	0.060	0.120	0.150
k	-1.08567	-1.11796	-1.2865	-0.942494	-0.903676
$\ln\delta$	-2.303	-2.526	-2.813	-2.120	-1.897
$\ln\lvert k\rvert$	0.092	0.112	0.252	-0.059	-0.101
Z	0.758	0.826	0.925	0.706	0.647

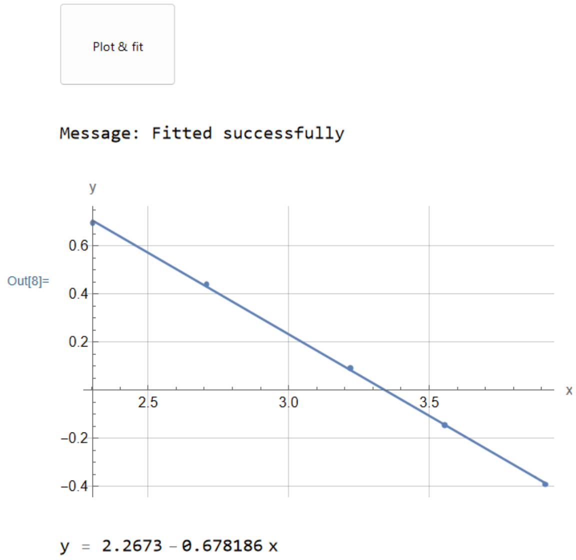
D_{gb}/D	25	35	50	15	10
k	-1.08567	-0.865926	-0.675888	-1.55123	-2.00586
$\ln[D_{gb}/D]$	3.219	3.555	3.912	2.708	2.303
$\ln\lvert k\rvert$	0.092	-0.144	-0.392	0.439	0.696
Z	0.758	0.603	0.473	1.072	1.413

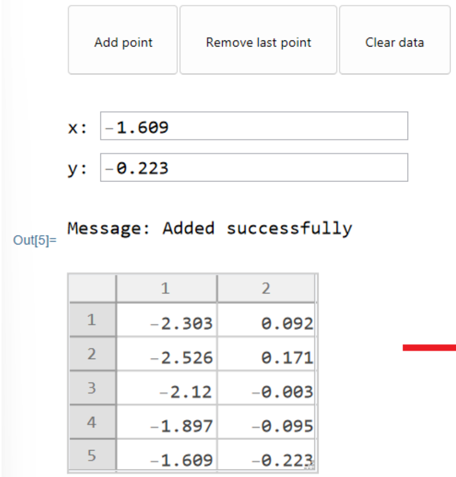
t	0.100	0.080	0.120	0.150	0.200
k	-1.08567	-1.18623	-0.99746	-0.908966	-0.799996
$\ln t$	-2.303	-2.526	-2.120	-1.897	-1.609
$\ln\lvert k\rvert$	0.092	0.171	-0.003	-0.095	-0.223
Z	0.758	0.836	0.699	0.634	0.558



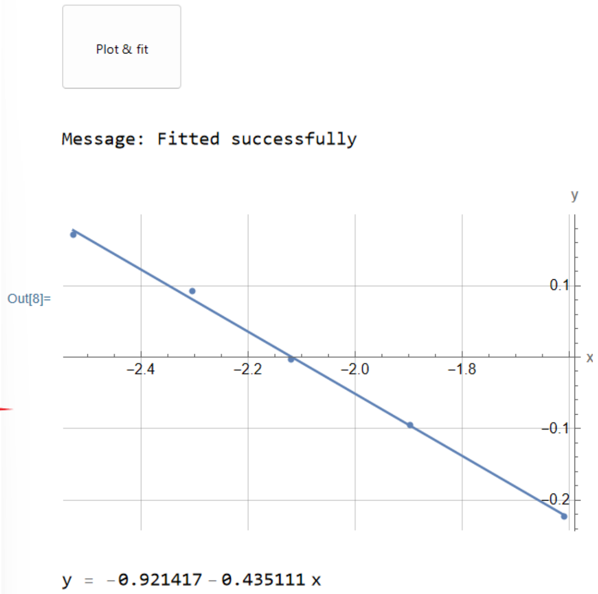


	1	2
1	3.219	0.092
2	3.555	−0.144
3	3.912	−0.392
4	2.708	0.439
5	2.303	0.696





	1	2
1	−2.303	0.092
2	−2.526	0.171
3	−2.12	−0.003
4	−1.897	−0.095
5	−1.609	−0.223



Coefficients we are looking for are then just slopes of plotted lines, so:

$$\xi = -0.39 \qquad \eta = -0.68 \qquad \zeta = -0.44$$

C3 Find the value of C as accurately as possible.

Knowing ψ, ξ, η, and ζ, we can find C as a slope of dependency of $Y = |k|$ on $X = \delta^{-0.39}[G_{gb}/D]^{-0.68}t^{-0.44}$. Calculating X for points obtained in the previous task, we can make a plot with InteractiveBuffer.nb:

The slope is:

$$C = 1.42$$

Add point Remove last point Clear data

x: 0.558

y: 0.799996

Message: Added successfully

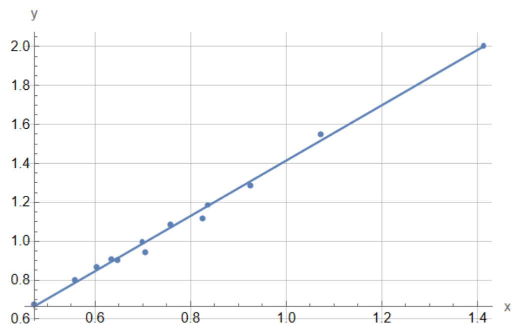
Out[5]=

	1	2
1	0.758	1.08567
2	0.826	1.11796
3	0.925	1.2865
4	0.706	0.942494
5	0.647	0.903676
6	0.603	0.865926
7	0.473	0.675888
8	1.072	1.55123
9	1.413	2.00586
10	0.836	1.18623
11	0.699	0.99746
12	0.634	0.908966
13	0.558	0.799996

Plot & fit

Message: Fitted successfully

Out[8]=



$y = -0.0081678 + 1.42207\,x$

T2. Muon anomalous magnetic moment

A1 For a muon in the beam find the difference between speed of light and muons' speed $\Delta v = c - v$ (provide formula and compute numerical value).

Energy of a relativistic particle can be expressed in terms of its momentum

$$E = \sqrt{p^2c^2 + m_\mu^2 c^4},$$

and velocity equals to

$$v = \frac{c^2 p}{E} = \frac{cp}{\sqrt{p^2 + m_\mu^2 c^2}}.$$

For the speed difference we have

$$\Delta v = c - v = c\left(1 - \frac{p}{\sqrt{p^2 + m_\mu^2 c^2}}\right) \approx \frac{1}{2}c\left(\frac{m_\mu c}{p}\right)^2 \approx 1.75 \cdot 10^5 \text{ m/s}.$$

For the given numerical values we can use the approximate formula, it gives deviation from the exact value only in the fourth figure.

A2 For a muon in the beam find the kinetic energy E_k, that is the difference between the full energy and the rest energy (provide formula and compute numerical value in Gev).

We use expression for the energy from the previous question. Note that for the given numerical values one can use approximate expression for the ultrarelativistic particle $E \approx pc$.

$$E_k = \sqrt{p^2c^2 + m_\mu^2 c^4} - m_\mu c^2 \approx 2.99 \text{ GeV}$$

A3 Find radius R of muon's orbit and period T of its motion. (Provide formulas and compute numerical values.)

From the relativistic equation of motion is

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} = \frac{q}{\gamma m_\mu}\vec{p} \times \vec{B}$$

it follows that the absolute values of velocity and momentum are constant . Therefore momentum rotates with

$$\frac{d\vec{p}}{dt} = \vec{\omega}_c \times \vec{p}, \quad \vec{\omega}_c = -\frac{q\vec{B}}{\gamma m}.$$

Here ω_c is muon angular velocity. The period is

$$T = \frac{2\pi}{\omega_c} = \frac{2\pi m\gamma}{qB} = 1.49 \cdot 10^{-7} \text{ s}.$$

The orbit radius is

$$R = \frac{vT}{2\pi} = \frac{\gamma m}{qB}v = \frac{\gamma m}{qB}\frac{pc^2}{E} = \frac{p}{qB} = 7.12 \text{ m}.$$

A4 Due to instability a muon can orbit the circle only finite time. Find number of revolutions N the muon makes before its decay.

Due to relativistic time dilation the muon lifetime in laboratory reference frame is $t = \gamma\tau$, therefore the number of revolutions is

$$N = \frac{t}{T} = \gamma\tau\frac{qB}{2\pi m_\mu \gamma} = \frac{qB\tau}{2\pi m_\mu} = 431.$$

A5	Obtain a formula for the spin time derivative, $d\vec{s}/dt$, express the answer in terms of $\vec{B}$, $\vec{s}$, g_μ and fundamental constants.	

Torque acting on a magnetic dipole in magnetic field is

$$\vec{M} = \vec{\mu} \times \vec{B},$$

thus equation of motion are

$$\frac{d\vec{s}}{dt} = \vec{M} = \vec{\mu} \times \vec{B} = g_\mu \frac{q}{2m_\mu} \vec{s} \times \vec{B} = -g_\mu \frac{q\vec{B}}{2m_\mu} \times \vec{s}.$$

A6	Find the z component of angular velocity ω_z of muon's spin. Express the answer in terms of B, g_μ and fundamental constants.	

From previous question it follows that the angular velocity is

$$\vec{\omega} = -g_\mu \frac{q\vec{B}}{2m_\mu}.$$

Therefore z component of angular velocity is

$$\omega_z = -g_\mu \frac{qB}{2m_\mu}.$$

B1	Find the x and y components of velocity $\vec{v}_C$ of reference frame C with respect to the reference frame A. Express the answer in terms of v_x, v'_y.	

Recall the formula for the adding velocity from the task. In our case $v'_x = 0$ (frame C velocity component along x axis), therefore denominator is $1 + v'xv_x/c^2 = 1$.

$$v_{Cx} = v_x, \quad v_{Cy} = v'_y\sqrt{1 - v_x^2/c^2} = v'_y/\gamma.$$

B2	Find the x'' and y'' components of velocity $\vec{v}''_A$ of the reference frame A with respect to the reference frame C. Take into account that v''_y is small.	

Note that if frame B moves relatively to A with velocity $\vec{v}$, then fram A moves relatively frame B with velocity $-\vec{v}$. Therefore frame B velocity with respect to frame C is $-v''_y$, and frame A velocity with respect to frame B is $-v_x$. The γ coefficient is calculated with velocity v'_y:

$$\gamma = \frac{1}{\sqrt{1 - v_y'^2/c^2}} \approx 1,$$

since we have to calculate only terms livear in v'_y. The velocities of frames A and B are orthogonal and the denominator in velocity addiction formula is equal to 1. The velocity of frame A with respect to frame C is

$$v''_{Ax} = -v_x, \quad v''_{Ay} = -v'_y.$$

B3	It turns out that $\vec{v}''_A \neq -\vec{v}_C$. It can be attributed to the fact that axes of the reference frame C are rotated with respect to the axes of frame A. Therefore all vectors in frame C are also rotated with respect to the frame A. Calculate the angle of rotation $\Delta\theta$ defined as an angle between vectors $-\vec{v}''_A$ and $\vec{v}_C$. The angle is positive if the rotation from first vector to the second vector viewed from the z axis is counterclockwise. (Axes x, y, z form right-handed coordinate system.) Express your answer in terms of v_x, v'_y, $\gamma = 1/\sqrt{1 - v_x^2/c^2}$, c.	**0.6**

The angle between two vector can be calculated from their vector product (length of vector product is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\alpha$, where α is the angle between the vector). Let $\vec{e}_x$ be an unit vector in x direction, $\vec{e}_y$ – in y direction, $\vec{e}_z$ – in z direction. The vector product is

$$-\vec{v}''_A \times \vec{v}_C = (v_x\vec{e}_x + v'_y\vec{e}_y) \times (v_x\vec{e}_x + (v'_y/\gamma)\vec{e}_y) = v_x v'_y \left(\frac{1}{\gamma} - 1\right)\vec{e}_z.$$

Both vectors are parallel to xy plane, therefore the vector product is collinear with z axis, positive z component corresponds to the positive angle of rotation. Up to the terms linear in v'_y we can approximate $|\vec{v}''_A| = |\vec{v}_C| = v_x$, thus angle of rotation is

$$\Delta\theta = \frac{1}{v_x^2} v_x v'_y \left(\frac{1}{\gamma} - 1\right) = -\frac{v'_y}{v_x}\left(1 - \frac{1}{\gamma}\right).$$

B4 Let a particle move with velocity $\vec{v}$ along x axis with respect to the frame A. Particle's acceleration is directed along the y axis, its projection is a_y. Frame B is the comoving frame for the particle at the considered moment (i.e. the velocity of the particle in it is zero) and moves with the speed v along x axis. After time interval dt (measured in laboratory reference frame) particle velocity is $\vec{v} + d\vec{v}$, corresponding comoving reference frame is C. From the previous question it follows that the axes of comoving frame rotate. Using results of the previous quaestion, find z component ω_T (angular velocity of Thomas precession) of the comoving frame angular velocity (in this case rotation is around z axis). Express the answer in terms of v_x, a_y, γ.

After time dt in laboratory frame of reference the particle velocity changes by $v_y = a_y dt$, in the frame B, which is comoving for the particle in the initial moment of time, corresponding velocity is $v'_y = \gamma v_y$. This result can be obtained from B1, or directly from velocity addiction formula (laboratory frame of reference moves relatively to frame B with velocity $-v_x$, particle velocity with respect to the laboratory frame of reference is $\vec{v} = (v_x, v_y)$):

$$v'_y = \frac{v_y\sqrt{1 - v_x^2/c^2}}{1 - v_x^2/c^2} = \frac{v_y}{\sqrt{1 - v_x^2/c^2}} = \gamma v_y.$$

Thus the rotation angle

$$d\theta = -\frac{\gamma a_y dt}{v_x}\left(1 - \frac{1}{\gamma}\right),$$

and angular velocity

$$\omega_T = \frac{d\theta}{dt} = -\frac{a_y}{v_x}(\gamma - 1).$$

Note that according our sign conventions positive angle of rotation corresponds to positive z component of the angular velocity.

B5 Let the muon move in homogeneous magnetic field B directed along z axis. Muon velocity is perpendicular to the magnetic field. Calculate the z component of Thomes precession angular velocity. Express the answer in terms of B, q, m_μ, γ.

From the equation of motion in magnetic field

$$\mu\frac{d\vec{p}}{dt} = m_\mu \frac{d(\gamma\vec{v})}{dt} = q\vec{v}\times\vec{B}$$

we calculate the acceleration component (note that γ is constant)

$$a_y = -\frac{q v_x B}{\gamma m_\mu},$$

thus angular velocity is

$$\omega_T = \frac{qB}{\gamma m_\mu}(\gamma - 1).$$

C1 Find the expression for z component of muon spin precession ω_s angular velocity with respect to laboratory reference frame. Express the answer in terms of B, g_μ, q, m_μ, γ.

The full angular velocity of precession is sum of the angular velocity of precession in magnetic field and Thomas precession angular velocity:

$$\omega_s = \omega_z + \omega_T = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{\gamma m_\mu}(\gamma - 1)$$

C2 In practice it is more convenient to consider spin rotation with respect to the direction of muon momentum. Find the angular velocity ω_a of muon spin rotation with respect to the direction of its momentum (that is also equal to the time derivative of the angle between momentum and spin). Express the answer in terms of B, a_μ, q, m_μ, γ.

The muon momentum rotates with anguler velocity which is equal to the angular velocity of the muon in magnetic field ω_c, which was calculated in A3, its z component is

$$\omega_z = -\frac{qB}{\gamma m_\mu}$$

Therefore the angular velocity of the spin with respect to the momentum is equal to the difference between the angular velocities of the spin and momentum (all rotations are arond z axis):

$$\omega_a = \omega_s - \omega_c = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{\gamma m_\mu}(\gamma - 1) + \frac{qB}{\gamma m_\mu} = -g_\mu \frac{qB}{2m_\mu} + \frac{qB}{m_\mu}.$$

By the relation $g_\mu = 2(1 + a_\mu)$, we obtain

$$\omega_a = -a_\mu \frac{qB}{m_\mu}.$$

C3 Experimentally measured frequency of muon spin precession with respect to the momentum direction is $f_a = 229081$ Hz. Find the expression for muon anomalous magnetic moment a_μ. Express the answer in terms of f_a, B, q, m_μ, γ. Compute the numerical value of a_μ.

Frequency in terms of angular velocity is given by $f_a = \omega_a/2\pi$, therefore

$$a_\mu = \frac{m_\mu \omega_a}{qB} = \frac{2\pi f_a m_\mu}{qB} = 0.00117.$$

The direction of the spin precession is not given, therefore we cannot determine the sign of a_μ, it turns out that $a_\mu > 0$.

T3. Frequency measurements with ring resonators

A1 What are the coefficients C_1 and C_2?

They are equal to 0 because the field cannot tend to infinity at a large distance from the fiber.

$$C_1 = 0$$
$$C_2 = 0$$

A2 Express k_{in} and k_{out} in terms of $\varepsilon, \omega, \beta$ and speed of light in vacuum $c = 1/\sqrt{\varepsilon_0\mu_0}$, using the wave equation.

Substituting solution type into the wave equation:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + (\omega^2\varepsilon/c^2 - \beta^2)A = 0$$

Considering that $\frac{\partial^2 A}{\partial x^2} \ll \frac{\partial^2 A}{\partial y^2}$, we get:

$$k_{in} = \sqrt{\frac{\omega^2\varepsilon(\omega)}{c^2} - \beta^2}$$
$$k_{out} = \sqrt{\beta^2 - \frac{\omega^2}{c^2}}$$

A3 Prove that the tangential components of the electric field strength are equal on both sides of the interface (as in electrostatics).

We can write Faraday's law of electromagnetic induction for a rectangular circuit with vertices at points $(a/2 - \delta a, y, z), (a/2 - \delta a, y, z + \Delta z), (a/2 + \delta a, y, z), (a/2 + \delta a, y, z)$.At $\delta a \to 0$, the derivative of the magnetic field flux tends to 0, and the magnitude of the circulation of the electric field strength remains constant. From this we can conclude that it is also equal to 0. At $\delta a \to 0$ the circulation can be expressed as $\Gamma = (E_{\tau 1} - E_{\tau 2})\Delta z = 0$, hence $E_{\tau 1} = E_{\tau 2}$.

A4 Write the boundary conditions for the tangential component of the electric field. Express B in terms of a, k_{in}, k_{out}.

$$B = \exp(k_{out}a/2)\cos(k_{in}a/2)$$

A5 Prove that the tangential components of the magnetic field strength are equal on both sides of the interface (as in magnetostatics).

The proof is similar to A3, but instead of the law of electromagnetic induction, the circulation theorem for the magnetic field strength is used.

A6 Prove that

$$H_z = E_0 \,\mathrm{Re}\left(\frac{i}{\mu_0\omega}\frac{\partial A}{\partial x}\exp(i(\omega t - \beta z))\right).$$

Using the differential form of Faraday law we get:

$$-\mu_0\frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

The derivative of H_z with respect to time can be expressed as:

$$\frac{\partial H_z}{\partial t} = i\omega H_z$$

$$H_z = E_0 \,\mathrm{Re}\left(\frac{i}{\mu_0\omega}\frac{\partial A}{\partial x}\exp(i\omega t - i\beta z)\right)$$

A7 Write the boundary conditions for the tangential component of the magnetic field $\vec{H}$. The answer can be expressed in terms of k_{in}, k_{out}, a.

To find H_z, differentiate A by x:

$$\frac{\partial A}{\partial x} = -k_{in}\sin(k_{in}x), |x| < a/2$$

$$\frac{\partial A}{\partial x} = -k_{out}\exp(k_{out}(-a/2+x))\cos(k_{in}a/2), |x| > a/2$$

Equating H_z on both sides of the interface:

$$\tan(k_{in}a/2) = \frac{k_{out}}{k_{in}}$$

A8 Substitute the values k_{in}, k_{out}, obtained in A2 into the equation from A7. Obtain an equation from which βcan be determined (this equation is solved numerically only). The equation can include $\beta, \omega, \varepsilon, c, a$.

$$\tan\left(\frac{a}{2}\sqrt{\frac{\omega^2\varepsilon(\omega)}{c^2} - \beta^2}\right) = \sqrt{\frac{\beta^2c^2-\omega^2}{\omega^2\varepsilon(\omega)-\beta^2c^2}}$$

B1 Assuming that there is no energy loss in the divider, find the relationship between r_s and t_s.

$$r_s^2 + t_s^2 = 1$$

B2 Express the field amplitude at the input to Q_1 of the splitter $B_{in}(t)$ in terms of κ and the field amplitude at the output of Q_2 at time $(t-\tau(\omega))$ — $B_{out}(t-\tau(\omega))$.

$$B_{in}(t) = \kappa B_{out}(t-\tau(\omega))$$

B3 Using the stationarity conditions, express $B_{in}(t)$ in terms of $B_{out}(t), \kappa, \omega, \tau$.

Let the φ be:

$$\varphi = -\omega\tau$$

Then

$$B_{out}(t) = it_sA_{in}(t) - r_sB_{in}(t)$$

$$B_{in}(t) = \kappa e^{i\varphi}B_{out}(t)$$

$$B_{in}(t) = \kappa e^{-i\omega\tau}B_{out}(t)$$

B4 Express $B_{in}(t)$ in terms of $A_0, r_s, t_s, \kappa, \omega, \tau$ and t, using the result of the previous task.

$$B_{in}(t) = \kappa e^{i\varphi}(it_sA_{in}(t) - r_sB_{in}(t))$$

$$B_{in}(t) = \frac{it_s\kappa e^{i\varphi}A_{in}(t)}{1+r_s\kappa e^{i\varphi}}$$

$$B_{in}(t) = \frac{it_s\kappa e^{-i\omega\tau}A_0e^{i\omega t}}{1+r_s\kappa e^{-i\omega\tau}}$$

B5 What is the power N_2, leaving the channel P_2? Express the answer in terms of $\omega\tau(\omega), \kappa, r_s$ and the power N_1 in channel P_1.

$$A_{out}(t) = -r_s A_{in}(t) + it_s B_{in}(t) = -A_{in}(t)\left(r_s + \frac{t_s^2 \kappa e^{i\varphi}}{1 + r_s \kappa e^{i\varphi}}\right)$$

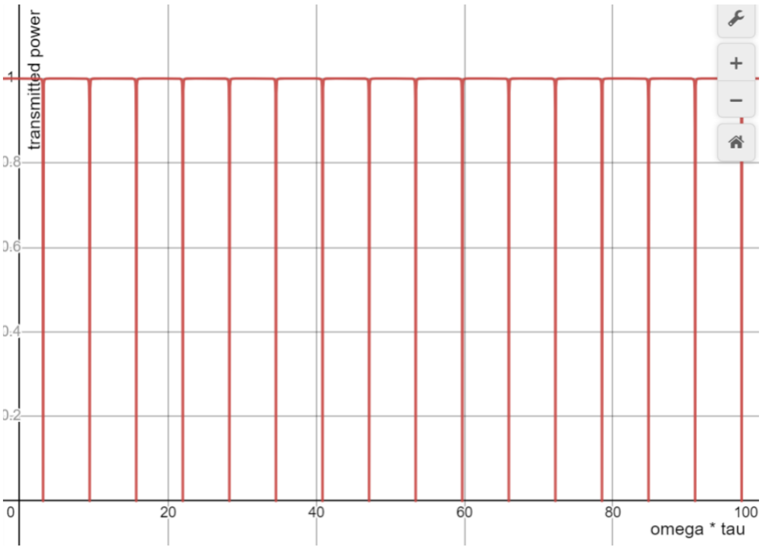
Let us reduce the right-hand expression to a common denominator and divide the square of the numerator modulus by the square of the denominator modulus:

$$\eta = \frac{N_2}{N_1} = \frac{\kappa^2 \sin^2(\omega\tau) + (r_s + \kappa\cos(\omega\tau))^2}{\kappa^2 r_s^2 \sin^2(\omega\tau) + (1 + r_s\kappa\cos(\omega\tau))^2}$$

B6 Sketch a qualitative plot of $N_2/N_1(\omega\tau)$ for fiber resonator with the following parameters:

- $\kappa = 1 - 5 \cdot 10^{-3}$;
- $t_s = 0.1$.

At what values of $\omega\tau$ is the ratio N_2/N_1 minimal?



$$\omega_{res}\tau = \pi(1 + 2n)$$

B7 Find the sharpness Q of the absorption peak with number $n = 100$.
Sharpness is the ratio of the peak frequency to the width of the region of frequencies for which the transmission dip is not less than half of the maximum dip of a particular peak.

Let us find the depth of the absorption peak by expanding the expression for the power ratio into a Taylor series:

$$\eta(\omega_{res}) \approx \frac{(1 - \kappa - t^2/2)^2}{(1 - \kappa + t^2/2)^2} \approx 0$$

$$\Delta\eta_{max} = 1 - \eta(\omega_{res}) \approx 1$$

Or by substituting the value φ from the previous paragraph, without putting it in a row:

$$\eta(\omega_{res}) \approx 1.6 \cdot 10^{-6} \approx 0$$

$$\Delta\eta_{max} \approx 1$$

Sharpness is the ratio of ω to the width of the region in which $\Delta\eta > \Delta\eta_{max}/2$. Let's estimate the width of this area:

$$\varphi = \delta + \pi(1 + 2n), \delta \ll \pi$$

Let us expand the Maclaurin series in terms of δ:

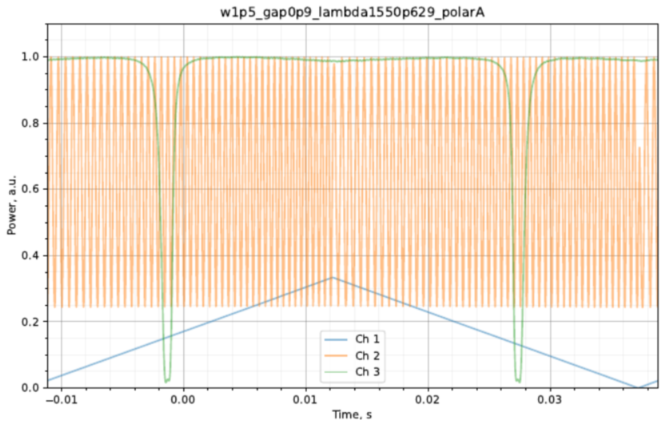
$$\eta \approx \frac{\delta^2 + (1-\kappa - t^2/2 + \delta^2/2)^2}{\delta^2 + (1-\kappa + t^2/2 + \delta^2/2)^2} \approx \frac{\delta^2}{\delta^2 + (1-\kappa + t^2/2)^2} = \frac{1}{2}$$

From here we get:

$$\delta \approx 1 - \kappa + t^2/2 \approx 0.01$$

$$Q = \frac{\omega_{res}}{\Delta\omega} = \frac{\pi(1+2n)}{2\delta} \approx \frac{\pi n}{1-\kappa+t^2/2} = \pi \cdot 10^4 = 3.14 \cdot 10^4$$

C1 Sketch a qualitative plot of the oscilloscope readings if it is known for sure that the frequency of the tunable laser reaches exactly one of the frequencies given in B6 (these are the frequencies where the FR transmittance is minimal).

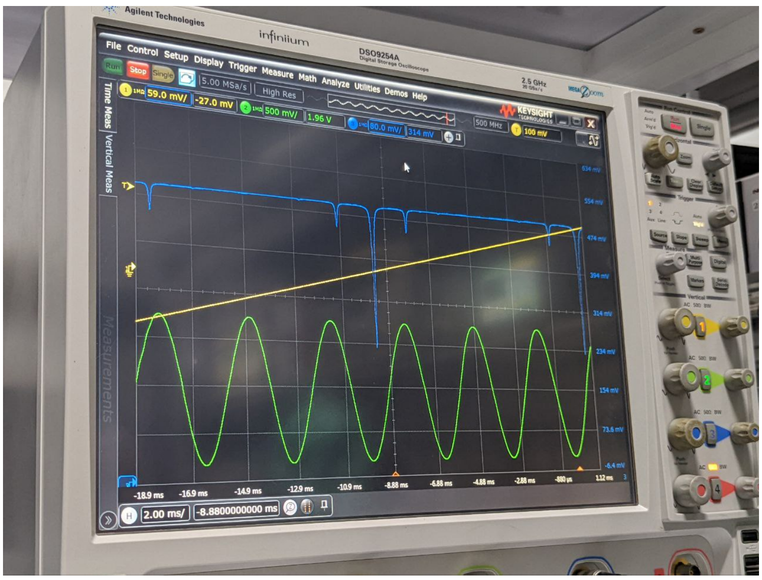


C2 Draw what the oscilloscope will show when $\Omega \approx 220 MHz$. Note that $\alpha \ll \Omega\omega_0$.

The field amplitude at the exit from the EOM will be equal to

$$E_{EOM}(t) = f(t)\cos(\omega(t)t) = \beta\cos(\omega t) + \frac{1-\beta}{2}\Big(\cos(\Omega+\omega)t + \cos(-\Omega+\omega)t\Big)$$

where $\omega = \omega(t) = \omega_0 + \alpha t$. In this case, three waves fall on the FR, the frequencies of which are shifted relative to each other by exactly Ω. Therefore, the number of absorption peaks will now triple: they can be observed at frequencies $\omega_{res}, \omega_{res}+\Omega, \omega_{res}-\Omega$, where ω_{res} is frequency corresponding to the minimum transmission (defined in B6).



C3 Estimate with what relative accuracy can the period ω_{MZI} be measured on this setup if the maximum signal frequency of the high-frequency oscillator is $\Omega_{max} = 1250MHz$?

We can measure one period by changing Ω in a way, so that the absorption maxima were separated by ω_{max} from each other. Then the absolute error would be of order $\omega_0/Q = 24MHz$, and relative $\omega_0/(Q\Omega_{max}) = 0.02$

D1 Express the group velocity v_g in terms of β_1.

$$v_g = 1/\beta_1$$

D2 Let us find the form of the soliton. The solution of NLSE can be found in the form:

$$F(z,s) = \frac{F_0 \exp(i\sigma z)}{\cosh(\theta s)}.$$

Experess F_0 и σ in terms of θ, β_1, β_2 и γ.

Let's calculate the derivatives and substitute them into the equation (reducing by $iF_0 \exp(i\sigma z)$):

$$\frac{\sigma}{\cosh\theta s} + \frac{\beta_2\theta^2}{2\beta_1^2}\Big(-\frac{2}{\cosh^3\theta s} + \frac{1}{\cosh\theta s}\Big) = \frac{\gamma F_0^2}{\cosh^3\theta s}$$

From here we get:

$$F_0^2 = -\frac{\beta_2\theta^2}{\gamma\beta_1^2}$$

$$\sigma = -\frac{\beta_2\theta^2}{2\beta_1^2}$$

D3 Express D_1 and D_2 in terms of β_1, β_2 and loop length L. *Hint: the BP natural frequency criterion: $B_{in}(t)$ and $B_{in}(t-\tau(\omega_\mu))$ have the same phase.*

Using the result of point B3, we get:

$$\beta(\omega(\mu))L + \pi = 2\pi(\mu + \mu_0)$$

Now let's substitute the expression for $\beta(\omega)$, and into it $\omega(\mu)$. We get:

$$\Big(\beta_0 + \beta_1(D_1\mu + \frac{D_2\mu^2}{2}) + \frac{\beta_2 D_1^2\mu^2}{2}\Big)L \approx 2\pi(\mu + \mu_0 + 1/2)$$

Equating the coefficients of μ and μ^2 on the right and left sides:

$$D_1 = \frac{2\pi}{\beta_1 L}$$

$$D_2 = -\frac{\beta_2 D_1^2}{\beta_1} = -\frac{4\pi^2\beta_2}{\beta_1^3 L^2}$$

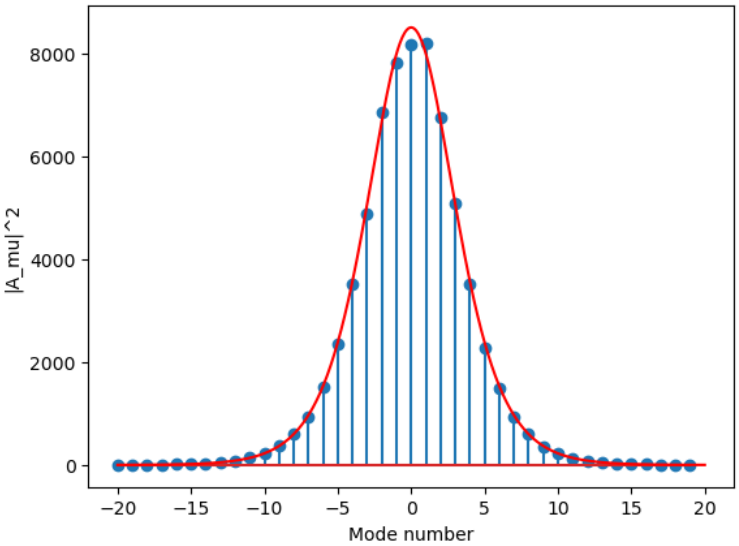
D4 Let the resonator be made of a material with $\chi > 0$. At what D_2 can solitons exist in it?

$F_0^2 > 0$ so $\beta_2 < 0$ and $D_2 > 0$

$$D_2 > 0$$

D5 Let a soliton with carrier frequency ω_0. circulates in the FR described in D3. The external laser does not work. Plot the emission spectrum of the resonator (the dependence of specific power on frequency) **qualitatively** in the frequency range $(\omega_0 - 20D_1, \omega_0 + 20D_1)$. Consider that $\omega_0/Q(\omega_0) \ll D_1$. (Remember that Q -is sharpness defined in B7)

Spectrum consists of narrow bands corresponding to the natural frequencies of the FR as shown on the figure. The envelope can be fitted by $F_1/\cosh^2(F_2(\omega - \omega_0))$ (here F_1, F_2 are some constants depending on F_0, σ, θ), but analytical expression for envelope form is not required.



D6 Estimate the absolute error of the angular frequency measurement ω using the spectrum from item D5.

We get a ruler with a pitch of D_1, and the absolute error is about a half of this pitch – $D_1/2$.

$$\Delta\omega \approx \frac{D_1}{2}$$

D7 Express the round-trip time τ_s through D_1.

$$\tau_s = \frac{2\pi}{D_1}$$